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When Constraints Interact: A Caution About Reference Variables, Identification Constraints, and Scale Dependencies in Structural Equation Modeling

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In traditional approaches to structural equations modeling, variances of latent endogenous variables cannot be specified or constrained directly and, consequently, are not identified, unless certain precautions are taken. The usual method for achieving identification has been to fix one factor loading for each endogenous latent variable at unity. An alternative approach is to fix variances using newer constrained estimation algorithms. This article examines the philosophy behind such constraints and shows how their appropriate use is neither as straightforward nor as noncontroversial as portrayed in textbooks and computer manuals. The constraints on latent variable variances can interact with other model constraints to interfere with the testing of certain kinds of hypotheses and can yield incorrect standardized solutions with some popular software.

Structural equation modeling programs are capable of analyzing a wide range of different models and techniques. In describing how to analyze such models, many textbooks advise the reader that to establish identification of a structural model, there must be one factor loading fixed at unity for each endogenous latent variable in the model. The observed variable with this unit loading identification (ULI) constraint is often referred to as a *reference variable*. The clear implication of the advice is that ULI constraints can, and should, be used almost automatically.

In this article, I demonstrate that this apparent rule regarding ULI constraints has exceptions and that applying the rule without understanding the exceptions can lead to errors in practice. I start with the basics—analyzing why ULI constraints are used, demonstrating some statistical consequences of their use, and developing some simple rules for determining when ULI constraints are not being used in the standard manner. Then I demonstrate, with some striking examples, how the traditional use of ULI constraints may lead to questionable or incorrect statistical conclusions when model coefficients are tested for equality using a chi-square difference test. Finally, I discuss potential solutions for the problems.

Notation

Different structural equation modeling programs use different models and notation systems. In this article, I use a slightly augmented version of the standard LISREL model and notation as described by Jöreskog and Sörbom (1989). Figure 1 represents a typical structural equation model with LISREL notation. Because the LISREL model contains so many model matrices, the standard notation uses repeated names. For example, there are two Θ matrices, Θ_δ and Θ_ϵ. To avoid the use of superscripts or triple subscripts, I refer, where the full LISREL model is discussed, to the elements of Θ_δ as $\delta_{i,j}$ and the elements of Λ_x as $\nu_{i,j}$, rather than using a more cumbersome notation like $\theta^{\delta}_{i,j}$ and $\lambda^{x}_{i,j}$. In some examples, I detach the measurement model for the Y variables from a larger path diagram and examine it in isolation. When the variances and covariances of the endogenous latent variables in η are discussed, I follow the example of Jöreskog and Sörbom (1989, p. 147) and use the notation Ω for the covariance matrix and $\omega_{i,j}$ for its elements. For all other elements of matrices I use a double-subscripted lowercase Greek letter corre-

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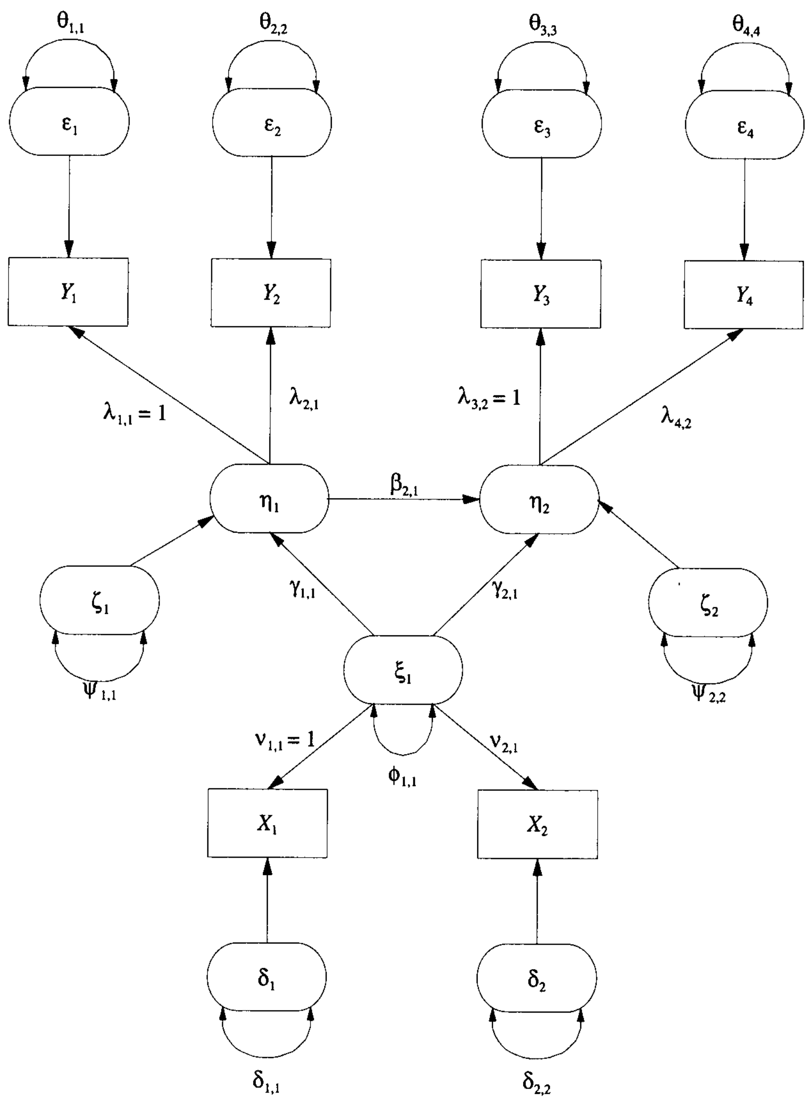

Figure 1. A typical structural equation model, with one exogenous latent variable and two endogenous latent variables, each with two indicators. Latent variables are shown in ovals, and observed variables are shown in rectangles. Standard LISREL notation is used. Paths without explicit coefficients have fixed values of 1. "= 1" means that a parameter has been assigned a fixed value of 1 to establish identification.

sponding to the matrix name. Thus, for example, free parameter elements of the standard LISREL matrix Φ are $\phi_{i,j}$, and free parameter elements in Λ_y are $\lambda_{i,j}$. Latent variables are shown in ovals, and manifest variables are shown in rectangles. Directed paths are shown with single-headed arrows. Undirected paths, representing covariances or variances, are shown with lines having two arrowheads.

Structural equation models usually are composed from several identifiable submodels. For example, in the model in Figure 1, there are two factor analysis *measurement models* at the top and bottom of the diagram, sandwiched around a multiple regression *structural model* in the center. An *exogenous variable* in a path diagram has no unidirectional arrow pointing to it. An *endogenous variable* has at least one unidirectional arrow pointing to it. At the bottom of Figure 1, there is a measurement model for the exogenous latent variable ξ_1. ξ_1 is a common factor of the two manifest variables X_1 and X_2. At the top of Figure 1 is a measurement model for the endogenous latent variables η_1 and η_2. In the center of the diagram is a structural equation model relating ξ_1, η_1, and η_2.

There are situations in which more than one set of coefficients will reproduce the observed data equally well. The coefficients in a structural equation model are *identified* if and only if there is only one set of coefficients that reproduces the data optimally. In the case of the present model, some coefficients are constrained to be equal to 1.0, that is, have ULI constraints, in order to make the other coefficients in the model identifiable. These coefficients (e.g., $\lambda_{1,1}$) with ULI constraints are indicated in the diagram with the notation "= 1" following the coefficient.

Characteristics of Properly Defined ULI Constraints

This section reviews the characteristics of properly applied ULI constraints in more detail than is found in most textbooks. Suppose we isolate the upper measurement model, involving η_1, η_2, Y_1 through Y_4, and ϵ_1 through ϵ_4, from Figure 1. In analyzing such a diagram, we use a "pipeline" metaphor. Imagine standing at any point in the diagram, and monitoring the numbers being "piped" through the paths. Doubling the standard deviation (or quadrupling the variance) of a variable simply doubles the magnitude of every number coming out of it. Path coefficients in such diagrams act like multipliers, so any number is multiplied by a path coefficient it passes through. Because every number passing through a path is multiplied by its path coefficient, the standard deviation of the number is multiplied by the absolute value of the coefficient, and the variance by the square of the coefficient. With these simple notions in tow, we note first that latent variable η_1 is never observed, and so its variability may only be inferred from two sources: (a) the variances and covariances of the variables with paths leading to η_1 and (b) the values of the path coefficients leading to η_1.

The variances of η_1 and η_2 are not uniquely defined and are free to vary unless some constraints are imposed on the free parameters in Figure 1. To see why, suppose that the ULI constraints were removed from $\lambda_{1,1}$ and $\lambda_{2,1}$ and that, by some combination of circumstances, the paths leading to η_1 and η_2 had values that caused η_1 to have a variance of 1. Suppose further that under these circumstances, the values .6, .3, .6, and .3 for parameters $\lambda_{1,1}$, $\lambda_{2,1}$, $\lambda_{3,2}$, and $\lambda_{4,2}$ lead to an optimal fit of the model to the data. Next, imagine we wished the variance of η_1 to be some value other than 1, say, 4. Quadrupling a variable's variance can be accomplished by doubling its standard deviation, or doubling every value of the variable. To achieve this, while maintaining the identical numbers arriving at Y_1, Y_2, and η_2 from η_1, we need only double all values on paths leading to η_1 while halving all values ($\lambda_{1,1}$, $\lambda_{2,1}$, and $\beta_{2,1}$) on paths leading away from η_1. Every number emerging from η_1 is doubled but is "passed through" coefficients that are now exactly half what they were. Thus, the numbers emerging at Y_1, Y_2, and η_2 are the same as they were. Because $\psi_{1,1}$ and $\gamma_{1,1}$ are free parameters that are attached to unidirectional paths, we can alter them (to halve the values of all numbers arriving at η_1) without affecting anything in the lower portion of the diagram in Figure 1.

The situation is demonstrated numerically in the two path diagram segments in Figure 2. The models in the upper and lower diagrams have identical fit to the data. However, we have manipulated the path coefficients to make the standard deviation of η_1 twice as large in the lower model. Any positive value for $\omega_{1,1}$, the variance of η_1, can be accommodated without affecting the overall fit of the model by simply adjusting the values of the path coefficients in the model. If some additional constraint is not placed on the values of these coefficients, then infinitely many sets of values will all reproduce the data equally well, and the parameter estimates will not be identified.

For exactly the same reason that we are able to set the variance of η_1 to any positive value we please, we also have the option of fixing the value of either $\lambda_{1,1}$ or $\lambda_{2,1}$ to any value we please. Suppose one path leading from η_1 is fixed. For example, suppose a ULI constraint is applied to the path coefficient $\lambda_{1,1}$, thus fixing it at 1.0, and now allowing it to be manipulated. What effects will this have? We can describe several. The model coefficients connected to η_1 will now be identified. The model will still fit exactly as well as it did before, as long as other model parameters are varied to compensate for the change in $\lambda_{1,1}$. Because $\lambda_{1,1}$ was changed from .6 to 1.0, the variance of η_1 will now be modified, and identified at a value (.36 in this case, corresponding to a change in the standard deviation from 1 to .6) that, in general, will not be some convenient, simple number (like 1.0).

This brief example typifies the way ULI constraints are supposed to work in practice. Let us summarize these properties.

1. When a ULI constraint is applied to a parameter, the primary goal is simply to establish identifica-

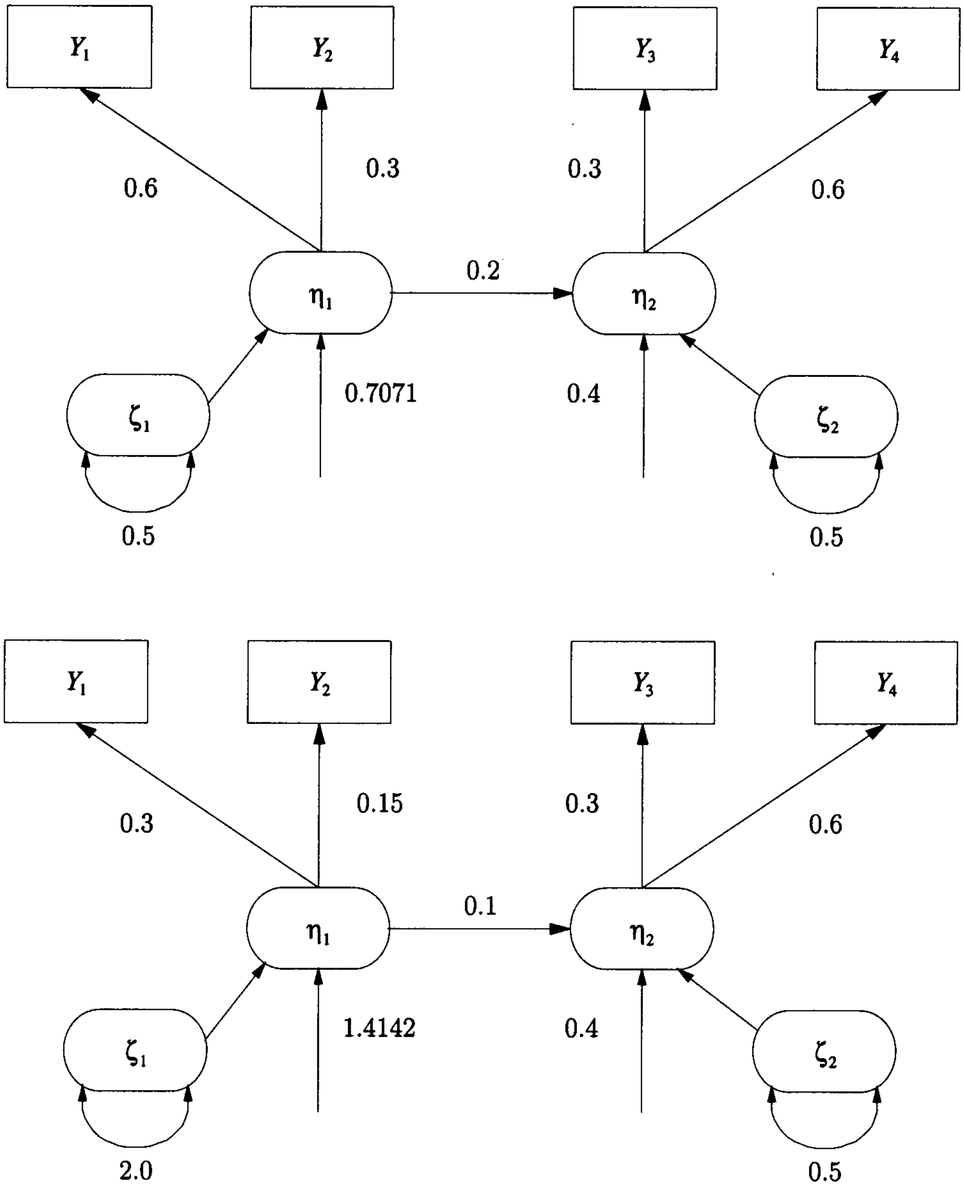

Figure 2. Doubling the standard deviation of a latent variable, and compensating by halving model parameters. Latent variables are shown in ovals, and observed variables are shown in rectangles. Standard LISREL notation is used. In the lower diagram, all paths leading to η_1 have been doubled, and the effect compensated for by halving all paths leading from η_1. All values arriving at Y_1 through Y_4 remain unchanged.

tion, and the precise value that the parameter is fixed to will not affect the fit of the model. Specifically, one could use the value 2.0 instead of 1.0, and the test statistic for the model would remain the same, because the fit of the model is invariant under change of scale of its latent variables.

2. The particular manifest variable chosen for the ULI constraint for any latent variable should not affect model fit. In the present example, fit will be the same if we constrain either $\lambda_{1,1}$ or $\lambda_{2,1}$ (but not both).
3. Path coefficients leading from a latent variable have the same relative magnitude regardless of the fixed value used in a ULI. Their absolute magnitude will go up or down depending on the fixed value used in the ULI. Thus, for example, if one

changes the 1.0 to a fixed value of 2.0, all path coefficients leading from the latent variable will double.

4. Any multiplicative change in the ULI constraint applied to a path coefficient will be mirrored by a corresponding division of the standard deviation of the latent variable the path leads from and by a corresponding division of path coefficients leading to the latent variable.

The above properties reflect the way ULI constraints are supposed to work in practice. The constraints are intended to be essentially arbitrary values imposed solely to achieve identification and are not intended to have any substantive impact on model fit or model interpretation.

There seems to be some confusion in the literature about the latter point. Numerous sources (e.g., Kline, 1998, p. 204) have made a statement to the effect that a ULI constraint for the loading of a particular manifest variable fixes the scale of the latent variable to be the same as the manifest variable. This misconception has led to the use of the term *reference variable* to refer to the manifest variable with the ULI attached. This view is wrong—if a value of unity is used, the variance of the latent variable is fixed to the variance of the *common part* of the manifest variable that has the ULI constraint. Moreover, as we have already seen, all other loadings emanating from the latent variable move up or down in concert with the value selected for the ULI constraint, and the variance of the common part is itself determined by the choice of variables in the measurement model. The key issue here is that residual variance includes error variance and unique variance, so fixing the metric of the latent variable to an observed variable's common variance has dubious value.

With these goals in mind, it seems reasonable to ask which hypotheses are invariant under choice of ULI constraints (or equivalently, under a choice of the scale of the latent variable) and which are not. Unless a particular choice of constraint (or latent variable variance) has a specific substantive meaning, a hypothesis that is not invariant under a choice of constraints will be difficult if not impossible to interpret.

For example, is the hypothesis that $\lambda_{1,1}$ equals $\lambda_{2,1}$ in the model of Figure 1 invariant under a change of scale of the latent variables? From the preceding analysis, it would seem that the answer is yes, because any change in the ULI constraint would be reflected proportionally in coefficients $\lambda_{1,1}$ and $\lambda_{2,1}$. The choice of the particular value used in the identifying constraint has no effect on this hypothesis. Another way of putting it is that the particular value of the variance of η_1 has no effect on the truth or falsity of the hypothesis. Similarly, the hypothesis that $\lambda_{3,2}$ and $\lambda_{4,2}$ are equal is invariant under choice of the fixed value used in an identifying constraint on the variance of η_2. We have established there are hypotheses about the model coefficients that are invariant under the choice of value we fix latent variable variances to, so long as the constraints are only to achieve identification. It seems reasonable to suggest that, if a hypothesis is invariant under the choice of the fixed value used in the identifying constraint, then the hypothesis might be considered meaningful when the value of 1.0 typically used in the ULI is used.

Some hypotheses are not invariant under a choice of the fixed value used in the identifying constraint. For example, in connection with the model in Figure 1, consider the hypothesis

$$H_0: \lambda_{2,1} = \lambda_{4,2}.$$

This hypothesis is not invariant under the choice of fixed value used in the identifying constraint on $\lambda_{1,1}$. Doubling the fixed value of $\lambda_{1,1}$ doubles the value of $\lambda_{2,1}$ while leaving $\lambda_{4,2}$ unchanged. In this case, the hypothesis is not invariant under change of scale of the latent variables.

In analyzing whether a ULI constraint (or set of constraints) is truly arbitrary, we should ask the following questions:

1. Does the goodness-of-fit statistic remain invariant under the choice of fixed value used in the identifying constraint? That is, if we change the 1.0 to some other number, does the value remain constant?
2. Does the goodness-of-fit statistic remain invariant under the choice of which manifest variable is the reference variable?
3. Do the relative sizes of path coefficients leading to the latent variable remain invariant under the choice of the fixed value used in the identifying constraint?
4. Do the relative sizes of path coefficients leading from the latent variable remain invariant under the choice of the numerical value used in the identifying constraint?
5. Does the choice of manifest variable to which the ULI constraint is applied affect the fit of the model?

In the next section, we develop some algebraic tools to enhance our understanding of how ULI constraints operate.

Algebraic Tools for Analyzing Model Constraints

In this section, we develop some important tools for studying simple structural models and understanding how ULI constraints operate. Recall the upper measurement model in Figure 1. Suppose we were fitting a model such as this in isolation to a set of data. That is, suppose we were simply fitting a confirmatory factor model with two correlated factors, each loading on only two variables. This model is shown in Figure 3.

The model, as shown, has two ULI constraints, as $\lambda_{1,1}$ and $\lambda_{3,2}$ have been assigned fixed values of 1. Using the pipeline metaphor, it is easy to see that without these constraints, the model would not be identified. For example, if $\lambda_{1,1}$ were not fixed, one could quadruple the variance of η_1 and compensate for it by halving $\omega_{1,2}$, $\lambda_{1,1}$, and $\lambda_{2,1}$.

One can, alternatively, identify the model by leaving $\lambda_{1,1}$ and $\lambda_{3,2}$ free and instead constraining the variances of η_1 and η_2 directly. All structural modeling programs allow the variances of exogenous latent variables to be fixed, but only a few allow the variances of endogenous latent variables to be fixed directly and conveniently. Thus, although we have the option of setting $\lambda_{i,j}$ or $\omega_{i,i}$ to 1.0 to identify the model when it is examined in isolation, it is much more common to see $\lambda_{1,1}$ and $\lambda_{3,2}$ fixed at 1.0 when the measurement model with endogenous factors is embedded in a LISREL model.

It is possible, with a simple model such as the one in Figure 3, to establish algebraically whether the model is identified and which population covariance matrices can fit the model. For example, in this case we establish, using algebraic analysis, that (a) the model is identified, and (b) only covariance matrices for which $\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}$ will fit the model.

Item (b) above is a constraint on the elements $\sigma_{i,j}$ of the population covariance matrix Σ that is implied by

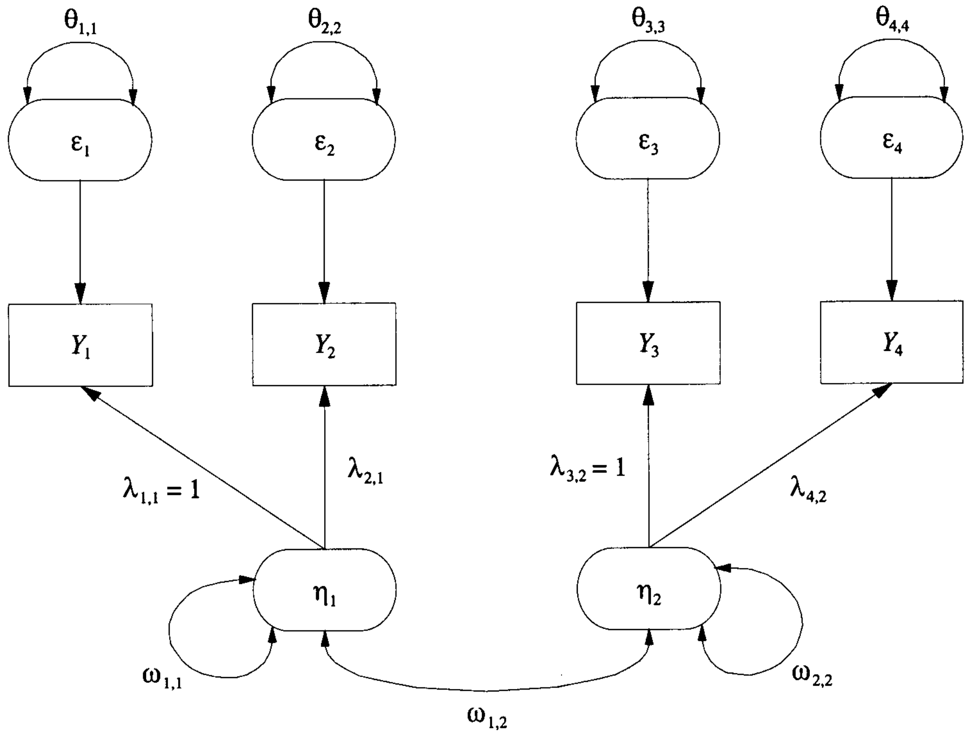

Figure 3. Confirmatory factor model with two correlated factors and two unit loading identification constraints to establish identification. Latent variables are shown in ovals, and observed variables are shown in rectangles. Standard LISREL notation is used. Paths without explicit coefficients have fixed values of 1. "= 1" means that a parameter has been assigned a fixed value of 1 to establish identification.

the model. It is an equation that can be reexpressed in the general form $f(\Sigma) = 0$, where $f(\Sigma)$ represents a scalar function of the elements of the covariance matrix, by simply moving all nonzero elements to the left. We refer to these constraints as Σ *constraints.* Different models, as is shown below, can have identical Σ constraints, which means that they are empirically indistinguishable. We refer to models with identical Σ constraints as Σ *equivalent.*

Deriving the above Σ constraints for the model of Figure 3 is a tedious exercise in basic algebra. The major steps are as follows:

1. Create the LISREL model matrices, with symbolic model parameters inserted in appropriate positions.
2. Compute a symbolic form for Σ, using the LISREL model equations, so that each nonduplicated element of Σ is expressed as a function of model parameters.
3. If Σ is of order $p \times p$, it will have $q = p(p + 1)/2$ nonduplicated elements. Set these elements equal to the symbolic formulas generated in Step 2 above. This will create a set of q symbolic equations. In what follows, we refer to these as the *model equations.*
4. To generate the implied constraints on Σ, eliminate the free parameters from the system of equations.
5. In order for the population covariance matrix Σ to fit the model perfectly, the Σ constraints must be satisfied. To determine whether the model is identified, solve for the augmented system of equations represented by the model equations and the Σ constraint equations. If the model is identified, each parameter can be expressed as a function of the elements of Σ.

To demonstrate how this process works, I begin with a very simple example. Suppose we have three data points, a, b, and c, and we have a "model" that says these data points may be explained in terms of two parameters, x and y. There are three model equations, and they are

$$x + y = a, \quad x - y = b, \quad 2x = c.$$

Not all data sets a,b,c can fit this model. Only certain data sets that obey a certain restriction can. To discover what the restriction is, and thereby discover what the model actually implies about the data, we systematically eliminate the model parameters from the above set of equations.

We begin by eliminating x, by solving the third equation and substituting the result (i.e., $x = c/2$) in the first two equations. We are now left with only two equations and one parameter, that is,

$$c/2 - y = b, \quad c/2 + y = a.$$

Adding these two equations together, we now eliminate y and arrive at the constraint equation,

$$c = a + b.$$

Only data sets satisfying this equation can fit our model.

Adding this equation to our original three model equations, we now have a set of equations that expresses the relationship between parameters and data, given that the model fits the data. We solve this set of equations to see if it has a unique solution. If it does, the model is identified.

Substituting $a + b$ for c in our original model equations, it is rather easy to deduce that the unique solution is

$$x = \frac{a+b}{2} = \frac{c}{2}, \quad y = \frac{a-b}{2}.$$

The identical steps are taken to solve the confirmatory factor model in Figure 3, but the algebra is more complicated. For all but the simplest structural equation models, these steps are best accomplished by use of special symbolic algebra software such as Mathematica or Maple. However, as I demonstrate in a subsequent section, one may in practice substitute a simple numerical procedure for algebraic analysis and can almost arrive at the correct conclusions.

For Figure 3 the LISREL model equation is simplified to

$$\Sigma_{yy} = \Lambda_y \Omega \Lambda_y' + \Theta_\epsilon, \tag{1}$$

with

$$\Lambda_y = \begin{bmatrix} 1 & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{4,2} \end{bmatrix}, \Omega = \begin{bmatrix} \omega_{1,1} & \omega_{2,1} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix},$$

$$\Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}. \tag{2}$$

Recall that Ω is the covariance matrix for the endogenous latent variables η_1 and η_2. This matrix is not generally defined in the full LISREL model, because in that context the variances and covariances of endogenous latent variables are fully determined by the paths leading to them. It is required here because we have temporarily removed the measurement model from the larger model. Computing Equation 1, we find that the model equations (with redundant elements above the diagonal not shown) are

$$
\begin{aligned}
\sigma_{1,1} &= \theta_{1,1} + \omega_{1,1}, \\
\sigma_{2,1} &= \lambda_{2,1}\omega_{1,1}, \\
\sigma_{2,2} &= \theta_{2,2} + \lambda_{2,1}^2\omega_{1,1}, \\
\sigma_{3,1} &= \omega_{2,1}, \\
\sigma_{3,2} &= \lambda_{2,1}\omega_{2,1}, \\
\sigma_{3,3} &= \theta_{3,3} + \omega_{2,2}, \\
\sigma_{4,1} &= \lambda_{4,2}\omega_{2,1}, \\
\sigma_{4,2} &= \lambda_{2,1}\lambda_{4,2}\omega_{2,1}, \\
\sigma_{4,3} &= \lambda_{4,2}\omega_{2,2}, \\
\sigma_{4,4} &= \theta_{4,4} + \lambda_{4,2}^2\omega_{2,2}.
\end{aligned}
$$

There are 10 equations in nine unknowns. If we successively eliminate the nine unknowns, following the same technique used in the preceding three-parameter example, we end up with a single equation,

$$\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}. \tag{3}$$

This equation is the Σ constraint for the model whose model matrices are in Equation 2. Any covariance matrix satisfying that constraint will fit the model perfectly. (Note that some covariances matrices satisfying Equation 3 may yield improper parameter values—e.g., a negative value for $\theta_{1,1}$.)

Examination of a model's Σ constraints can reveal interesting aspects of the model. For example, we see that the variances of the four observed variables are not present in the Σ constraint for this model, and all subscript values occur equally often on both sides of the constraint equation. This implies that any change of scale of the four observed variables cannot affect whether the model fits Σ, and so in this case an equivalent constraint is $\rho_{3,1}\rho_{4,2} - \rho_{3,2}\rho_{4,1} = 0$. This equation has an interesting form. It is the difference of two products of correlations, each of which involves the same four variables but in different permutations. Spearman (1904) showed that all Σ constraints for an unrestricted single factor model could be expressed in this form. He called such a constraint a *tetrad equation* and the left side of the equation a *tetrad difference*. Early attempts at statistical testing for the single common factor model were based on examining the tetrad differences and determining whether they departed significantly from zero. Spearman (1927) discussed this approach and gave several formulas for the approximate standard error of a tetrad difference. More recently, interest in tetrad equations has been revived, both in the confirmatory testing of measurement models (Bollen & Ting, 1993, 1998, 2000) and in the development of computer algorithms to uncover causal structure (Glymour, Scheines, Spirtes, & Kelly, 1987). As we see below, not all Σ constraints are tetrad difference equations.

Adding Equation 3 to the original system, one can show that, if the data fit the model, and certain degenerate conditions (e.g., $\sigma_{4,1} = 0$) do not hold, then closed form solutions for all model parameters are available. For example,

$$\lambda_{4,2} = \frac{\sigma_{4,2}}{\sigma_{3,2}},$$

$$\lambda_{2,1} = \frac{\sigma_{4,2}}{\sigma_{4,1}}.$$

Suppose we perform the identical analysis on a slightly different version of the model in Figure 3. In this version, we remove the restrictions on $\lambda_{1,1}$ and $\lambda_{3,2}$ and leave them as free parameters to be estimated. Instead, we identify the variances of η_1 and η_2 directly by the restrictions $\omega_{1,1} = 1$ and $\omega_{2,2} = 1$.

The revised model has the following model matrices:

$$\Lambda_y = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & \lambda_{3,2} \\ 0 & \lambda_{4,2} \end{bmatrix}, \Omega = \begin{bmatrix} 1 & \omega_{2,1} \\ \omega_{2,1} & 1 \end{bmatrix}, \tag{4}$$

$$\Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}.$$

These matrices produce a different set of model equations from the first version of the model, but after eliminating the unknown parameters as before, we find the Σ constraint for this model is the same as Equation 3. On the other hand, when we add this constraint to the model equations and solve for the model parameters, we obtain different solutions for some model parameters. For example,

$$\lambda_{2,1} = \sqrt{\frac{\sigma_{2,1}\sigma_{4,2}}{\sigma_{4,1}}}, \tag{5}$$

$$\lambda_{4,2} = \sqrt{\frac{\sigma_{4,2}\sigma_{4,3}}{\sigma_{3,2}}}. \tag{6}$$

These results imply that any data fitting the "ULI constraint version" of the model will fit the "unit variances" version, but the parameter values will in general be different. Applying a ULI constraint for each factor does not usually fix the variance of that factor to 1. Rather, it fixes it to a value that is essentially arbitrary. The parameters common to both models will generally be different, because the second version of the model fixes the factor variances to 1, whereas the first version usually does not.

The two versions of the model in Figure 3 that are represented by the model matrices in Equations 2 and 4 represent the most commonly used method for identifying the factor variances in a confirmatory factor model. However, they are not the ways to accomplish this, and there are other models that are Σ equivalent and are also fully identified. We return to this fact later in the next section.

Interaction Between ULI Constraints and Equality Constraints

In structural equation modeling, a hypothesis of equality of path coefficients is usually tested with a chi-square difference test. Unfortunately, some chi-square difference tests do not perform in the intended manner—they are compromised by a phenomenon I call *constraint interaction.* To examine constraint interaction in a simple context, suppose one wished to test the hypothesis that $\lambda_{2,1}$ and $\lambda_{4,2}$ are equal in the simple factor model of Figure 3. The difference test would normally proceed by first fitting a version of this model with the two loadings constrained to be equal and then fitting the model without the equality constraint. The difference between the two chi-square statistics is a chi-square with 1 degree of freedom.

The precise parameterization of the constrained version of the model depends on how one chooses to identify the variances of the latent variables. The standard approach uses ULI constraints on $\lambda_{1,1}$ and $\lambda_{3,2}$ and replaces $\lambda_{4,2}$ with $\lambda_{2,1}$ in the model equations. An alternative approach is to constrain $\omega_{1,1}$ and $\omega_{3,3}$ to unity, while still replacing $\lambda_{4,2}$ with $\lambda_{2,1}$. This seems simple enough, and one might expect these two models to fit a covariance matrix equally well. However, they do not! If we pursue the steps illustrated in the previous section, we discover these two models have different Σ constraints.

The situation is summarized in Table 1, which presents the Σ constraint equations corresponding to each degree of freedom for the four models. (For simplicity of exposition, I give the most general reduced form of the Σ constraint equations that do not assume any elements of Σ are zero. It is assumed that denominators of the constraint equations are nonzero.) Comparing the constraint equations for the 2nd degree of freedom for the models where $\lambda_{2,1}$ and $\lambda_{4,2}$ are required to be equal, it becomes clear that these two models are not Σ equivalent, because the constraints are not the same. The Σ constraints for the 2nd degree of freedom differ depending on whether variances are identified by (a) using ULI constraints on $\lambda_{1,1}$ and $\lambda_{3,2}$ or (b) fixing the variances of η_1 and η_2 at unity. To demonstrate that the versions of the model with $\lambda_{2,1} = \lambda_{4,2}$ are actually different, we can use the Σ constraint equations in columns 3 and 5 of Table 1 to construct a population covariance matrix Σ that fits the model with ULI constraints perfectly but does not fit the model with unit variances perfectly. For example, consider the following:

$$\Sigma = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \\ 4 & & & \\ 0.9 & 4 & & \\ 0.8 & 0.5 & 1 & \\ 0.5 & 0.3125 & 0.3 & 1 \end{bmatrix}.$$

The reader may verify, using structural equation modeling software, that this matrix perfectly fits the following LISREL model using ULI constraints:

$$\Lambda_y = \begin{bmatrix} 1 & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{2,1} \end{bmatrix}, \Omega = \begin{bmatrix} \omega_{1,1} & \omega_{2,1} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix},$$
$$\Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}. \quad (7)$$

The reader may also verify that the above covariance matrix does not perfectly fit the corresponding model with unit variance latent variables:

$$\Lambda_y = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & \lambda_{3,2} \\ 0 & \lambda_{2,1} \end{bmatrix}, \Omega = \begin{bmatrix} 1 & \omega_{2,1} \\ \omega_{2,1} & 1 \end{bmatrix},$$
$$\Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}. \quad (8)$$

Table 1
Σ Constraints for Four Confirmatory Factor Models

	ULI constraints		Unit variance constraints	
df	Unrestricted	$\lambda_{2,1} = \lambda_{4,2}$	Unrestricted	$\lambda_{2,1} = \lambda_{4,2}$
1	$\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}$	$\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}$	$\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}$	$\sigma_{3,1}\sigma_{4,2} = \sigma_{3,2}\sigma_{4,1}$
2		$\sigma_{4,1} = \sigma_{3,2}$		$\sigma_{4,1} = \left(\dfrac{\sigma_{2,1}}{\sigma_{4,3}}\right)\sigma_{3,2}$

Note. The table presents Σ constraint equations for four variations of a confirmatory factor model with four observed variables and two correlated factors. Σ constraints are equations that the elements of Σ must satisfy for a model to fit the data perfectly. There is one constraint equation for each degree of freedom in the model. The two unrestricted models presented here have no equality restrictions on the factor loadings and use either ULI or unit variance constraints to identify latent variable variances. The other two models restrict $\lambda_{2,1}$ and $\lambda_{4,2}$ to be equal and use either ULI or unit variance constraints to identify latent variable variances. ULI = unit loading identification.

We can, just as easily, construct a covariance matrix that fits the model matrices of Equation 8 but not those of Equation 7. The fact that two methods of identifying the variances of η_1 and η_2 with this model produce empirically nonequivalent models when factor loadings are constrained to be equal was noticed, and commented on, by O'Brien and Reilly (1995). However, their article, an important conceptual contribution, did not carry through its algebraic analysis to the point where all implied constraints were expressed as Σ constraints, that is, equations expressed solely in terms of the elements of Σ. When models are reexpressed in terms of Σ constraints, it is much easier to see why they are not empirically equivalent, that is, exactly how and why a given Σ may fit one model and not another.

We have seen that two seemingly equivalent methods for identifying factor variances are not always equivalent. Without the equality constraint $\lambda_{2,1} = \lambda_{4,2}$ the two methods for fixing variances are equivalent. When this constraint is added, the two methods yield models that are not Σ equivalent. What this means, in turn, is that a chi-square difference test of the hypothesis that $\lambda_{2,1} = \lambda_{4,2}$ will produce different results, depending on whether the model is parameterized with ULI constraints or with standardized (unit variance) latent variables. Before discussing this phenomenon in more detail, we should digress briefly to note an important and apparently unnoticed fact that helps explain the source of the nonequivalency. When the model includes the constraint that $\lambda_{2,1} = \lambda_{4,2}$, only one ULI constraint (or, alternatively, unit variance constraint) is necessary to establish identification. For example, consider again the model of Equation 7. Although the frequently cited "rule" for identifying variables might lead one to believe that two ULI constraints are necessary to identify this model, it actually will remain identified if one of the ULI constraints is relaxed. Suppose we relax the constraint on $\lambda_{1,1}$ and allow it to be a free parameter. In this case, the model matrices become

$$\Lambda_y = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{2,1} \end{bmatrix}, \Omega = \begin{bmatrix} \omega_{1,1} & \omega_{2,1} \\ \omega_{2,1} & \omega_{2,2} \end{bmatrix},$$
$$\Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}. \quad (9)$$

If we construct the model equations and eliminate the parameters, we discover that the Σ constraint for the model of Equation 9 is the same as Equation 3. Thus, this model is Σ equivalent to the models of Equations 2 and 4.

This fact does not seem to have been noticed in previous discussions of structural modeling. For this model, unlike those examined previously, the constraint $\lambda_{2,1} = \lambda_{4,2}$ has an unintended side effect. Besides constraining the two factor loadings to be equal, it also identifies the variance of the first factor, η_1.

We can verify this informally, using the pipeline metaphor approach. Reexamine the model of Figure 3, and imagine that the ULI constraint on $\lambda_{1,1}$ has been replaced by an equality constraint, that is, $\lambda_{2,1}$ and $\lambda_{4,2}$ must remain equal, as in Figure 4. Imagine further that the variances of η_1 and η_2 are identified at some value and that the model fits perfectly. Now ask the question, "Can we vary the variance of either η_1 or η_2 and compensate for it by adjusting other model coefficients?" First imagine that we double the standard deviation of η_1. We could try to compensate for this by halving all coefficients attached to η_1 (i.e., $\lambda_{1,1}$ $\lambda_{2,1}$, and $\omega_{1,2}$). Note that halving the value of $\lambda_{2,1}$ would require halving the value of $\lambda_{4,2}$ because of the equality constraint. However, this cannot be done without changing the fit of the model.

We have discovered a surprising fact. The equality constraint on the λs not only constrains them to be equal but it also fixes the variance of η_1 to a particular value. What value? The value is essentially arbitrary, that is, it might be described as "whatever value occurs when $\lambda_{3,2}$ is fixed as 1, and $\lambda_{2,1}$ and $\lambda_{4,2}$ are constrained to be the same free parameter."

A similar result holds when unit variance constraints instead of ULI constraints are used. With $\lambda_{2,1}$ and $\lambda_{4,2}$ constrained to be equal, one need only constrain either $\omega_{1,1}$ or $\omega_{2,2}$ to unity to identify both variances.

Consequently, once the equality constraint on the λs is in place, the unnecessary second ULI (or unit variance) constraint actually overconstrains the model beyond what is necessary for identification. The effect of the unnecessary additional constraint depends on its type—adding the second ULI constraint forces parallel λs to be equal, whereas adding a second unit variance constraint forces the factor variances to be standardized.

Unfortunately, the chi-square difference test for equal λs cannot be performed unless the identification constraints are kept constant for the two tests, because if the unnecessary identification constraint is removed, the two models will have the same degrees of freedom and will not be nested.

The results discussed in this section have several important implications:

1. A chi-square difference test for equal factor loadings on different factors is not "scale-free," that is, it depends on the scaling of the factors involved.
2. If loadings on different factors are constrained to be equal, then the factor variances may be identified without a ULI constraint's being used on every factor.
3. Conditions 1 and 2 may generalize to many situa-

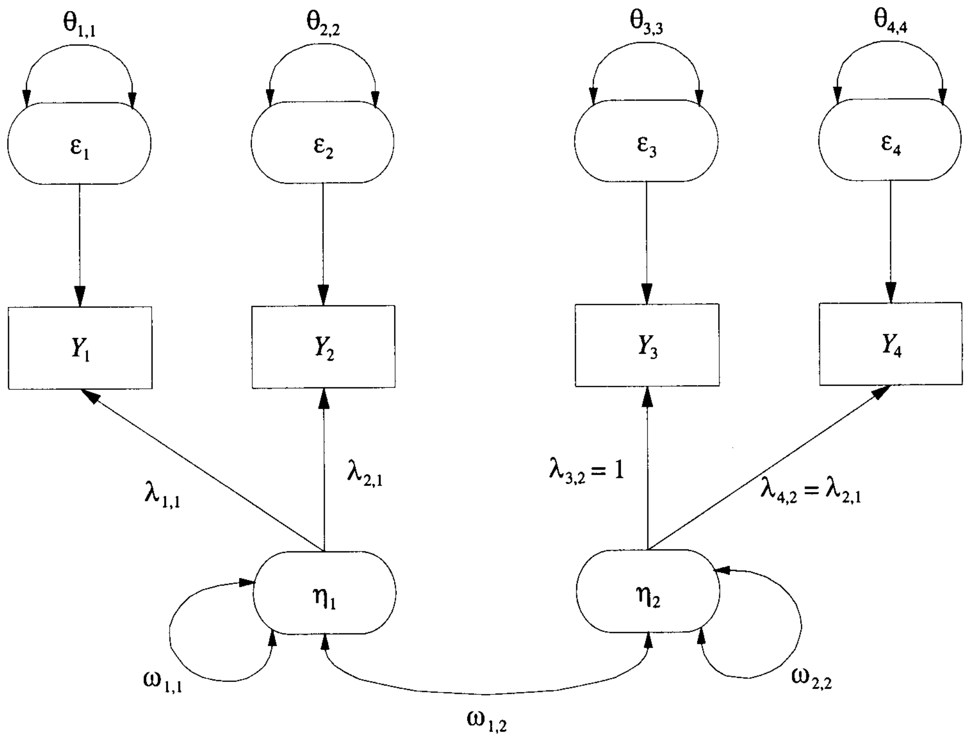

Figure 4. A confirmatory factor model with one unit loading identification constraint and one equality constraint, empirically equivalent to the model in Figure 3. Latent variables are shown in ovals, and observed variables are shown in rectangles. Standard LISREL notation is used. Paths without explicit coefficients have fixed values of 1. "= 1" means that a parameter has been assigned a fixed value of 1 to establish identification.

tions other than the simple one discussed here. They will certainly generalize to any situation that can be conceptualized as a factor model.
4. When the chi-square difference test is not scale invariant, choice of a particular scale might be based on substantive grounds. If no reasonable substantive grounds exist, then such a test may not be meaningful.

In the next section, we examine how the lessons learned in the context of a simple confirmatory factor analysis model generalize to more complex structural equation models.

Metric-Setting Effects in the General LISREL Model

The LISREL model includes two measurement models that are actually confirmatory factor models similar to the one we studied in the preceding section. It is not surprising that the metric-setting effects we saw in the preceding section will continue to hold if the factor model is embedded in a structural equation model, because the fit of the measurement model also affects the fit of the overall model.

For example, the four-variable, two-factor model was actually embedded in the larger model of Figure 1, so this latter model should also be sensitive to metric-setting effects if we try to test the restricted model that $\lambda_{2,1} = \lambda_{4,2}$. The model of Figure 1 corresponds to the following LISREL model equations:

$$\Lambda_x = \begin{bmatrix} 1 \\ \nu_{2,1} \end{bmatrix}, \Phi = [\phi_{1,1}], \Theta_\delta = \begin{bmatrix} \delta_{1,1} & 0 \\ 0 & \delta_{2,2} \end{bmatrix},$$
$$\Lambda_y = \begin{bmatrix} 1 & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{4,2} \end{bmatrix}, \Gamma = \begin{bmatrix} \gamma_{1,1} \\ \gamma_{2,1} \end{bmatrix},$$
$$\Psi = \begin{bmatrix} \psi_{1,1} & 0 \\ 0 & \psi_{2,2} \end{bmatrix}, \Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 \\ \beta_{2,1} & 0 \end{bmatrix}. \tag{10}$$

This model is parameterized with ULI constraints. We can fit this model, with or without the equality constraint $\lambda_{2,1} = \lambda_{4,2}$, using any structural equation modeling program. An alternative parameterization does not use ULI constraints and substitutes the restrictions that ξ_1, η_1, and η_2 all have unit variances. It uses the following model equations:

$$\Lambda_x = \begin{bmatrix} \nu_{1,1} \\ \nu_{2,1} \end{bmatrix}, \Phi = [1], \Theta_\delta = \begin{bmatrix} \delta_{1,1} & 0 \\ 0 & \delta_{2,2} \end{bmatrix},$$
$$\Lambda_y = \begin{bmatrix} \lambda_{1,1} & 0 \\ \lambda_{2,1} & 0 \\ 0 & \lambda_{3,2} \\ 0 & \lambda_{4,2} \end{bmatrix}, \Gamma = \begin{bmatrix} \gamma_{1,1} \\ \gamma_{2,1} \end{bmatrix},$$
$$\Psi = \begin{bmatrix} \psi_{1,1} & 0 \\ 0 & \psi_{2,2} \end{bmatrix}, \Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 \\ \beta_{2,1} & 0 \end{bmatrix}. \tag{11}$$

The model implied by these equations must be solved subject to the nonlinear constraints that imply unit variances for η_1 and η_2. With the aid of Mathematica, one may compute these constraints as

$$\omega_{1,1} = \sigma^2_{\eta_1} = \gamma^2_{1,1}\phi_{1,1} + \psi_{1,1} = 1,$$
$$\omega_{2,2} = \sigma^2_{\eta_2} = \beta_{2,1}\gamma_{1,1}\gamma_{2,1}\phi_{1,1} + \gamma^2_{2,1}\phi_{1,1} + \beta_{2,1}[\gamma_{1,1}\gamma_{2,1}\phi_{1,1} + \beta_{2,1}(\gamma^2_{1,1}\phi_{1,1} + \psi_{1,1})] + \psi_{2,2} = 1. \tag{12}$$

Table 2 presents Σ constraint equations[1] for the four models with ULI constraints and unit variance constraints, with and without the restriction that $\lambda_{2,1} = \lambda_{4,2}$. The table shows that, without the equality constraint on $\lambda_{2,1}$ and $\lambda_{4,2}$, both versions of the model have six Σ constraints corresponding to 6 degrees of freedom, and the constraint equations are the same. Consequently, the two versions of the model are empirically equivalent.

Table 2 also demonstrates that, when we constrain $\lambda_{2,1}$ and $\lambda_{4,2}$ to be equal, the models are no longer

[1] The actual list of Σ constraint equations produced directly by elimination is quite long. For brevity and simplicity of exposition, I give the most general nonredundant reduced form of the Σ constraint equations that guarantees perfect model fit without requiring any element of Σ to be precisely zero. This reduced equation set does not subsume certain covariance matrices (with precisely zero elements) that also fit the model.

Table 2
Σ *Constraints for Four Confirmatory Factor Models*

	ULI constraints		Unit variance constraints	
df	Unrestricted	$\lambda_{2,1} = \lambda_{4,2}$	Unrestricted	$\lambda_{2,1} = \lambda_{4,2}$
1	$\sigma_{3,1} = \frac{\sigma_{4,1}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,1} = \frac{\sigma_{4,1}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,1} = \frac{\sigma_{4,1}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,1} = \frac{\sigma_{4,1}\sigma_{6,3}}{\sigma_{6,4}}$
2	$\sigma_{3,2} = \frac{\sigma_{4,2}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,2} = \frac{\sigma_{4,2}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,2} = \frac{\sigma_{4,2}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{3,2} = \frac{\sigma_{4,2}\sigma_{6,3}}{\sigma_{6,4}}$
3	$\sigma_{5,1} = \frac{\sigma_{5,4}\sigma_{6,1}}{\sigma_{6,4}}$	$\sigma_{5,1} = \frac{\sigma_{5,4}\sigma_{6,1}}{\sigma_{6,4}}$	$\sigma_{5,1} = \frac{\sigma_{5,4}\sigma_{6,1}}{\sigma_{6,4}}$	$\sigma_{5,1} = \frac{\sigma_{5,4}\sigma_{6,1}}{\sigma_{6,4}}$
4	$\sigma_{5,2} = \frac{\sigma_{5,4}\sigma_{6,2}}{\sigma_{6,4}}$	$\sigma_{5,2} = \frac{\sigma_{5,4}\sigma_{6,2}}{\sigma_{6,4}}$	$\sigma_{5,2} = \frac{\sigma_{5,4}\sigma_{6,2}}{\sigma_{6,4}}$	$\sigma_{5,2} = \frac{\sigma_{5,4}\sigma_{6,2}}{\sigma_{6,4}}$
5	$\sigma_{5,3} = \frac{\sigma_{5,4}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{5,3} = \frac{\sigma_{5,4}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{5,3} = \frac{\sigma_{5,4}\sigma_{6,3}}{\sigma_{6,4}}$	$\sigma_{5,3} = \frac{\sigma_{5,4}\sigma_{6,3}}{\sigma_{6,4}}$
6	$\sigma_{6,2} = \frac{\sigma_{4,2}\sigma_{6,1}}{\sigma_{4,1}}$	$\sigma_{6,2} = \frac{\sigma_{4,2}\sigma_{6,1}}{\sigma_{4,1}}$	$\sigma_{6,2} = \frac{\sigma_{4,2}\sigma_{6,1}}{\sigma_{4,1}}$	$\sigma_{6,2} = \frac{\sigma_{4,2}\sigma_{6,1}}{\sigma_{4,1}}$
7		$\sigma_{4,1} = \sigma_{3,2}$		$\sigma_{4,1} = \left(\frac{\sigma_{2,1}}{\sigma_{4,3}}\right)\sigma_{3,2}$

Note. The table presents Σ constraint equations for four variations of a structural equation model shown in Figure 1. Σ constraints are equations that the elements of Σ must satisfy for a model to fit the data perfectly. There is one constraint equation for each degree of freedom in the model. The two unrestricted models presented here have no equality restrictions on the factor loadings and use either ULI or unit variance constraints to identify latent variable variances. The other two models restrict $\lambda_{2,1}$ and $\lambda_{4,2}$ to be equal and use either ULI or unit variance constraints to identify latent variable variances. ULI = unit loading identification.

empirically equivalent, because the Σ constraints corresponding to the 7th degree of freedom are different. This difference occurs because, as in the case with the simple confirmatory factor model, only one of the ULI or unit variance constraints is necessary to establish identification. With the equality constraint on the λs in place, the superfluous ULI (or unit variance constraint) now functions to constrain the model over and above that which is required for identification.

We have seen how, when a measurement model with equality constraints on loadings for different factors is embedded in a structural equation model, there can be important consequences: (a) A chi-square difference test for equal loadings on different factors requires constraints on variances of both factors; (b) the choice of a method for fixing the variance of latent variables will change the model that is implied; and (c) consequently, the difference test is not invariant under changes of the scale of the latent variables, and any choice of scaling method must be justified on a substantive basis.

The astute reader may, at this point, be asking the obvious question, "Because the relative size of factor loadings is often of no great interest in structural equation modeling, is this phenomenon worthy of much consideration?" It is therefore important to recognize that the problem described above may generalize to other situations.

For example, reconsider the model of Figure 1. Suppose one wished to test whether the direct effect of ξ_1 on η_1 is equal to its effect on η_2, namely, that $\gamma_{1,1}$ and $\gamma_{2,1}$ are equal. The traditional approach would be to first fit the model without the equality constraint on the γ coefficients and then fit it with the constraint. The difference between the two chi-square test statistics would be used to perform a chi-square difference test.

However, algebraic analysis similar to that performed above establishes that a model with an equality constraint on $\gamma_{1,1}$ and $\gamma_{2,1}$ and two identification constraints (so that a chi-square difference test can be performed) will have a different Σ constraint for one of its degrees of freedom, depending on whether ULI constraints or unit variance constraints are used to identify the variances of η_1 and η_2.

Equation 13 gives a covariance matrix that perfectly fits the model of Figure 1 with $\gamma_{1,1}$ and $\gamma_{2,1}$ constrained to be equal and the ULI constraints replaced by unit variance constraints:

$$\Sigma = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 & X_1 & X_2 \\ 0.400000 \\ 0.090000 & 0.400000 \\ 0.050000 & 0.060000 & 0.400000 \\ 0.060000 & 0.072000 & 0.124026 & 0.400000 \\ 0.058333 & 0.070000 & 0.077778 & 0.093333 & 0.400000 \\ 0.050000 & 0.060000 & 0.066667 & 0.080000 & 0.090000 & 0.400000 \end{bmatrix}. \tag{13}$$

Computer software—for example, Mx (Neale, 2002; Neale, Boker, Xie, & Maes, 1999), RAMONA (part of Systat; Browne & Mels, 1999), or SEPATH (part of Statistica; Steiger, 1995)—that produces a fully standardized solution with constrained estimation will generate a standardized solution that fits the matrix in Equation 13 perfectly. It is extremely important to realize that some commercial computer software will not necessarily find this perfect solution for the following reasons. Recall, first, that when the γ coefficients are constrained to be equal, and two variance constraints are used, the model is not, in general, invariant under change of scale of its latent variables. If these data are analyzed with a model having ULI constraints, in general the fit will not be perfect. This, in turn, implies that the standardized solution produced by some programs will not be correct, nor will fit be perfect. For example, versions of LISREL (up to 8.3, at least) produce a standardized solution by first finding the unstandardized solution with ULI constraints and then transforming it to standardized form. This procedure assumes, implicitly, that the ULI constraints function only to establish identification, but when the γ coefficients are constrained to be equal, the ULI constraints do not function as planned. Because fit is not perfect with ULI constraints, the resulting standardized solution will also not have perfect fit. In practice, this means that, when constraints interact, software that produces a standardized solution without using constrained estimation almost always will have the wrong chi-square value.

So far, our discussions of constraint interaction and metric-setting effects have relied heavily on two tools: (a) use of the pipeline metaphor for detection and (b) algebraic analysis of Σ constraints for verification and quantification. These methods have some drawbacks. The pipeline metaphor is simple but requires practice and careful use, and algebraic analysis of Σ constraints requires (except for the simplest models) facility with the use of a symbolic algebra program such as Mathematica. Unfortunately, if the model is reasonably complicated, symbolic algebra programs such as Mathematica may not solve for model equations in a reasonable time frame, so simpler methods are desirable. In the next section, a simple numerical method that will uncover the vast majority of modeling situations in which constraint interactions might be a problem is discussed.

A Simple Numerical Approach to Detecting Constraint Interaction

Constraint interaction, when it occurs, has an important implication—equality of two coefficients will not be invariant under changes of scale of the latent variables. Whether a hypothesis test "makes sense" under such circumstances is an issue to be addressed in a subsequent section. Here, we concentrate on a simple approach to detecting constraint interaction in practice.

Recall that, when constraints interact, a model that fits perfectly under one parameterization will not fit perfectly under another. This is because the model itself is not invariant under changes of scale of the latent variables. One way of testing whether a model is sensitive to the scale of the latent variables is to test whether the fit of the model is sensitive to the value used in the ULI constraint. Generally, of course, this value is set equal to 1. For models that are invariant under changes of scale of the latent variables, the magnitude of this value may be varied to any nonzero number without affecting model fit. However, for models that are not invariant under changes of scale, varying the magnitude of the value will affect model fit.

Consequently, a simple way of detecting constraint interaction is the following: (a) For the model with equality constraints, compute a chi-square fit statistic with the standard ULI constraints in place; (b) then, alter the model so that one of these constraints is altered, say, to a value of 2 instead of 1; (c) recompute the chi-square statistics; and (d) if the two chi-square statistics are not identical within rounding error, and convergence has occurred in each case, then the model is not invariant under changes of scale of the latent variables. As an example, consider the model of Figure 1, with ULI constraints, and with $\lambda_{2,1}$ and $\lambda_{4,2}$ constrained to be equal. This model has these LISREL model matrices:

$$
\Lambda_x = \begin{bmatrix} 1 \\ \nu_{2,1} \end{bmatrix}, \Phi = [\phi_{1,1}], \Theta_\delta = \begin{bmatrix} \delta_{1,1} & 0 \\ 0 & \delta_{2,2} \end{bmatrix},
$$
$$
\Lambda_y = \begin{bmatrix} 1 & 0 \\ \lambda_{2,1} & 0 \\ 0 & 1 \\ 0 & \lambda_{2,1} \end{bmatrix}, \Gamma = \begin{bmatrix} \gamma_{1,1} \\ \gamma_{2,1} \end{bmatrix},
$$
$$
\Psi = \begin{bmatrix} \psi_{1,1} & 0 \\ 0 & \psi_{2,2} \end{bmatrix}, \Theta_\epsilon = \begin{bmatrix} \theta_{1,1} & 0 & 0 & 0 \\ 0 & \theta_{2,2} & 0 & 0 \\ 0 & 0 & \theta_{3,3} & 0 \\ 0 & 0 & 0 & \theta_{4,4} \end{bmatrix}, \tag{14}
$$
$$
B = \begin{bmatrix} 0 & 0 \\ \beta_{2,1} & 0 \end{bmatrix}.
$$

Suppose we fit the covariance matrix of Equation 15 to the model of Equation 14.

$$
\Sigma = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 & X_1 & X_2 \\ 1.00 \\ 0.40 & 1.00 \\ 0.20 & 0.10 & 1.00 \\ 0.20 & 0.10 & 0.20 & 1.00 \\ 0.20 & 0.10 & 0.20 & 0.20 & 1.00 \\ 0.10 & 0.05 & 0.10 & 0.10 & 0.30 & 1.00 \end{bmatrix}. \tag{15}
$$

Using a "dummy" sample size of 100, we find that, fitting this model with LISREL or any other covariance structure program, the maximum-likelihood chi-square statistic has a value of 0.86 when the ULI constraints are their normal value of 1.0. Now, suppose we alter the model so that the ULI constrained value on $\lambda_{1,1}$ is changed from 1 to 2. Fitting this model to the data, we obtain a chi-square statistic of 0. The modified model fits perfectly! This verifies that the model is not invariant under changes of scale.

An Example From the LISREL Manual

One of the standard example models given in LISREL and other structural equation modeling computer manuals (Jöreskog & Sörbom, 1984, 1989) is the multiple indicators multiple causes (MIMIC) model of Duncan, Haller, and Portes (1968). Figure 5 is a path diagram for this model. This path diagram is equivalent to one given in the LISREL 7 manual (Jöreskog & Sörbom, 1989, p. 146). Jöreskog and Sörbom analyzed the algebraic properties of the model very extensively to show that the parameters are in fact identified. For our purposes, the key aspects of the diagram, and of the Jöreskog and Sörbom (1989) analysis, are as follows: (a) There are reciprocal paths with coefficients called β_{21} and β_{12} by Jöreskog and Sörbom from variable η_1 to η_2, and vice versa, and these are constrained to be equal; and (b) there are ULI constraints on the paths from η_1 to Y_1 and from η_2 to Y_4, ostensibly in order to identify the variances of η_1 and η_2.

Jöreskog and Sörbom (1989) performed a chi-square difference test of the hypothesis that $\beta_{2,1} = \beta_{1,2}$. Inspection of the model in Figure 5 using the techniques described above reveals that when the constraint $\beta_{2,1} = \beta_{1,2}$ is active, only one of the two ULI constraints is necessary to produce model identification. Adding a second ULI constraint overconstrains the model and, rather than leaving the discrepancy function unchanged, actually increases it. Consequently, the chi-square difference statistic calculated by Jöreskog and Sörbom is not invariant under change of scale of the latent variables.

Jöreskog and Sörbom (1989), though mindful (e.g., Cudeck, 1989) that treating the correlation matrix as if it were a covariance matrix can lead to erroneous estimates for parameter standard errors, analyzed the correlation matrix from Duncan et al. (1968) as if it were a covariance matrix for illustrative purposes. When the model of Figure 5 is fit to the correlation matrix, one obtains a chi-square value of 26.90, with 17 degrees of freedom. The parameter estimate for $\lambda_{2,1}$ is 1.0610. When we change the value of the fixed coefficient on the path from η_1 to Y_1 from 1.0 to 2.0 and fit the revised model to the data, we note two anomalies: (a) The chi-square value changes to 29.09, and (b) the value of the parameter estimate for $\lambda_{2,1}$ does not double. It changes from 1.0610 to 2.1066. If ULI constraints were being applied in the traditional fashion and were serving only to establish identification, the chi-square value would not change, and the value of the parameter estimate would double. Consequently, whether $\beta_{2,1}$ and $\beta_{1,2}$ are equal is sensitive to the scale of η_1 and η_2.

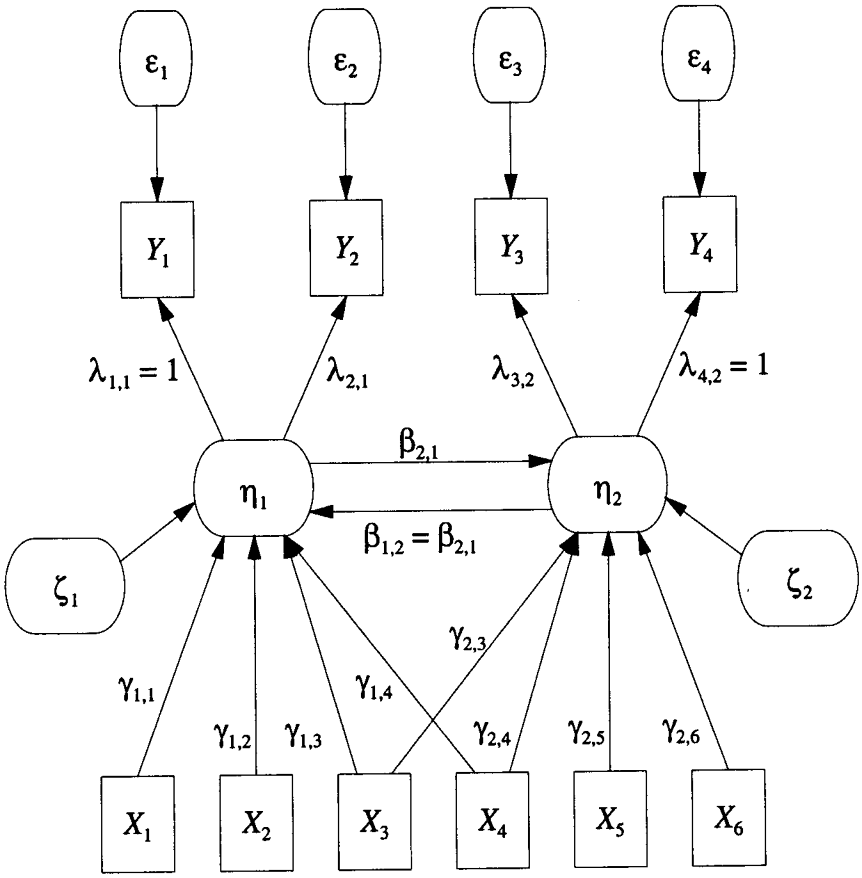

Figure 5. A recursive structural equation model, after Duncan, Haller, and Portes (1968). $\beta_{2,1}$ and $\beta_{1,2}$ are constrained to be equal. Latent variables are shown in ovals, and observed variables are shown in rectangles. Standard LISREL notation is used. Paths without explicit coefficients have fixed values of 1. "= 1" means that a parameter has been assigned a fixed value of 1 to establish identification.

Standardized Models as an Optimal Arbitrary Choice

In the preceding section, we learned that the question of whether $\beta_{2,1}$ and $\beta_{1,2}$ are equal is not scale invariant. One potential solution to the problem is to require the variances of η_1 and η_2 to be equal by testing a standardized model, that is, one in which all latent variables, both endogenous and exogenous, have unit variance. In this case, the null hypothesis is rephrased as one about the equality of standardized coefficients. In such a case, the standardization becomes part of the model.

The technique for obtaining standardized estimates without ULI constraints was discussed briefly by Mels (1989), and a general discussion of the statistical theory and methodology for fitting covariance structure models with nonlinear constraints was given by Browne and DuToit (1992). Because the variances of η_1 and η_2 are (nonlinear) functions of the model parameters, they can be fixed at 1.0 by estimating the model coefficients, subject to a (usually complicated) nonlinear constraint. One such constraint is required for each variance. Each constraint adds a degree of freedom to the model. In effect, one surrenders the right to "fix the metric" of the latent variables to that of the common parts of a particular manifest variable and instead adopts the convention of fixing the variance of the latent variable to 1.

Several software packages implement estimation of structural equation models subject to constraints. Some of the more convenient options are discussed in the next section.

It must be emphasized that, for the above solution to work, standardized estimates must be generated using a constrained estimation technique. Unfortunately, many software packages produce "standardized" models by a two-step procedure that involves (a) calculation of the unstandardized model, using ULI constraints to establish identification of the variances of η_1 and η_2, and (b) use of a simple algebraic transformation to compute the standardized estimates after the fact. This approach will not reliably solve our problem in the current situation. ULI constraints do not usually lead to unit variances for η_1 and η_2. Because the fit of the model is not invariant under change of scale, ULI constraints do not produce perfect fit when constrained variances do. Transforming the solution from unstandardized to standardized after the iterative fitting procedure doesn't change the fit. Ironically, then, one can have data that perfectly fit a standardized model, and software programs using the two-step standardized procedure will not detect the appropriate solution.

Software Implementation

In situations in which ULI constraints create problems, several software packages allow them to be eliminated by using constrained estimation to produce a standardized solution. The ease with which this can be done varies substantially across programs. Some programs, such as RAMONA and SEPATH, allow the variances of latent variables to be constrained automatically and directly as part of the language specification. These programs allow ULI constraints to be eliminated with virtually no effort. Mx, a freeware program by Neale, Boker, Xie, and Maes (1999), allows specification of complex constraints not only on parameters but on complex functions of model matrices. As a result, it is relatively easy for the experienced Mx user to produce a standardized solution using constrained estimation as discussed in Section 2.7 of the Mx manual. LISREL, while it allows nonlinear constraints on parameters, does not include the extensive matrix calculation facilities in Mx. Consequently, the LISREL user must be sophisticated enough to be able to write an expression for the variance of each endogenous latent variable as a scalar function of model parameters. Often, this will be nearly impossible without the use of symbolic algebra manipulation software and substantial expertise in statistical theory. On the other hand, it is relatively easy to use a program like Mathematica to derive these constraint equations and then transfer them to the LISREL command language.

Discussion and Recommendations

In this article, we have explored the use of ULI constraints and "reference variables," and we have discovered that their use is not quite as straightforward as represented in a number of textbooks and program manuals. Some of the key points that were discussed are as follows:

1. ULI constraints, though simple and convenient, are not the only method available for fixing the variance of latent variables. Several modern software packages using constrained estimation allow the variances of latent variables to be constrained directly.
2. In the majority of structural equation models, ULI constraints have no impact on overall model fit, and their only real purpose is to establish identification.
3. When path coefficients attached to two or more different latent variables are constrained to be equal (as part of a hypothesis test for equality), constraining the variance of one of the latent variables will usually constrain the variance of the others, because of a phenomenon I call *constraint interaction*. In such a situation, ULI constraints (or other variance-fixing devices) are no longer required on all latent variables. If they are used, then (a) the model fit will no longer be invariant under changes of scale of the latent variables, and (b) the hypothesis of equality of constraints will not be invariant under a choice of metric for the latent variables. This creates a dilemma for the practitioner. Equality constraints and variance constraints are necessary to perform the chi-square difference test, but unless there are substantive grounds for choosing a metric for the latent variables, the test would be of questionable validity.
4. In cases in which standardized latent variables can be justified, constrained estimation techniques can be used to test the hypothesis of equal standardized coefficients with a chi-square difference test. One cannot reliably use traditional two-step standardized solutions to test a hypothesis of equal standardized coefficients.
5. When constraint interaction occurs, and ULI constraints are used incorrectly, the outcome of the resulting significance test is, in a sense, an accident of fate.

Constraint interaction can be detected by following some simple procedures designed to detect whether the fit of a particular structural equation model is not invariant under changes of scale of its latent variables. First, for each identification constraint, vary the value of the fixed loading from the standard value of 1 to some other value, say 2. Next, check whether (a) the value of the chi-square statistic remains the same and (b) the relative sizes of the constrained loading and its companion loadings for the other indicators of the latent variable remain equal. If either check fails, then the "identification constraint" is not simply establishing identification—it is constraining the fit of the model to the data in some other way. Consequently, the fit of the model will not be invariant under changes of scale of the latent variables.

Once it is determined that fit of a model is not invariant under changes of scale of its latent variables, the natural question to ask is whether the model can meaningfully be tested. In situations in which standardized model coefficients can be compared in a meaningful way, the solution to this problem is to work with models in which the latent variables are constrained to have unit variance. Such models are not always reasonable, especially when more than one sample is being tested. However, in many circumstances, standardized coefficients are of interest, and their use circumvents the problems discussed here, provided they are produced by a constrained estimation approach. Some popular structural equation modeling programs are not designed to allow convenient generation of standardized estimates via nonlinear constraints. The freeware package Mx can perform the calculations with minimal programming, and the commercial programs RAMONA and SEPATH allow such estimation to be performed automatically by selecting a program option.

References

Bollen, K. A., & Ting, K. F. (1993). Confirmatory tetrad analysis. In P. V. Marsden (Ed.), *Sociological Methodology* (Vol. 3, pp. 147–175). Washington, DC: American Sociological Association.

Bollen, K. A., & Ting, K. F. (1998). Bootstrapping a test statistic for vanishing tetrads. *Sociological Methods and Research, 27,* 77–102.

Bollen, K. A., & Ting, K. F. (2000). A tetrad test for causal indicators. *Psychological Methods, 5,* 3–22.

Browne, M. W., & DuToit, S. H. C. (1992). Automated fitting of nonstandard models. *Multivariate Behavioral Research, 27,* 269–300.

Browne, M. W., & Mels, G. (1999). Path Analysis (RAMONA). In SYSTAT 10: Statistics II [Computer software and manual] (pp. II-233–II-291). Chicago: SPSS Inc.

Cudeck, R. (1989). Analysis of correlation matrices using covariance structure models. *Psychological Bulletin, 105,* 317–327.

Duncan, O., Haller, A. O., & Portes, A. (1968). Peer influence on aspirations: A reinterpretation. *American Journal of Sociology, 74,* 119–137.

Glymour, C., Scheines, R., Spirtes, P., & Kelly, K. (1987). *Discovering causal structure.* Orlando, FL: Academic Press.

Jöreskog, K. G., & Sörbom, D. (1984). Lisrel VI: Analysis of linear structural relationships by maximum likelihood, instrumental variables, and least squares methods. [Computer software manual]. Mooresville, IN: Scientific Software.

Jöreskog, K. G., & Sörbom, D. (1989). *Lisrel 7: A guide to the program and applications.* Chicago: SPSS.

Kline, R. B. (1998). *Principles and practice of structural equation modeling.* New York: Guilford Press.

Mels, G. (1989). *A general system for path analysis with latent variables.* Unpublished master's thesis, University of South Africa, Pretoria, South Africa.

Neale, M. C. (2002). Mx: Statistical Modeling (Version 1.52a) [Computer software]. Richmond, VA: Virginia Commonwealth University, Department of Psychiatry.

Neale, M. C., Boker, S. M., Xie, G., & Maes, H. H. (1999). Mx: Statistical Modeling (5th ed.) [Computer software manual]. Richmond, VA: Virginia Commonwealth University, Department of Psychiatry.

O'Brien, R. M., & Reilly, T. (1995). Equality in constraints and metric-setting measurement models. *Structural Equation Modeling, 2,* 1–12.

Spearman, C. (1904). General intelligence, objectively determined and measured. *American Journal of Psychology, 15,* 201–293.

Spearman, C. (1927). *The abilities of man.* London: Macmillan.

Steiger, J. H. (1995). Structural Equation Modeling (SEPATH). In *STATISTICA 5, Volume III* [Computer software and manual] (pp. 3539–3688). Tulsa, OK: StatSoft, Inc.

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