Comparing Correlations: Pattern Hypothesis Tests Between and/or Within Independent Samples

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Abstract

General procedures are presented for comparing correlations or groups of correlations between and/or within samples, with or without the assumption of multivariate normality. The methods are implemented in a freeware *Mathematica* package, WBCORR, and illustrated with numerical examples, including comparison of correlation matrices over time, simultaneous comparison of sets of correlations in two or more independent groups, comparison of several predictors of the same criterion, comparison of circumplex structures across groups, and comparison of predictor-criterion relations across groups.

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Samples

Preamble

Many years ago, a psychologist colleague approached me with a question about

how to compare two dependent correlations. He was puzzled because equations in two of

the few papers then available on the topic seemed to disagree (actually, both contained

minor typographical errors). I found the topic fascinating, and got deeply involved in

related research of my own. Reviewing the literature, I saw a paper by Rod McDonald

(1975) on testing pattern hypotheses on correlations, which discussed a general method

for comparing correlations, and alluded to the availability of some computer software he

had produced for performing the analyses At the time, Rod and I were engaged in a

spirited (to put it mildly) debate on factor indeterminacy, and it was with some

trepidation that I sent him a letter asking if I could acquire the software he had produced.

Rod responded quickly and graciously with the requested materials, which saved me a

substantial amount of time producing a Monte Carlo study I was working on (Steiger,

1980b). This is only one of the ways that my own work (like that of countless others) has

been influenced positively by interactions with Rod over the years.

Introduction

Normal Theory Pattern Hypothesis Tests

A *pattern hypothesis* on a set of statistical parameters specifies that sets of

parameters are equal to each other, or to specified numerical values. Pattern hypotheses

on elements of one or more correlation coefficients have wide application in the analysis

of social science data. Unfortunately, the general statistical theory necessary for such

comparisons was not made widely available in textbooks prior to 1970, and seemed generally unknown to social scientists at that time. A classic paper by Olkin and Siotani (1964, 1976) gave examples of several interesting comparisons, and several subsequent papers (Dunn & Clark, 1969; Neill & Dunn, 1975) by Dunn and her students raised awareness of the fact that correlations may be compared either within or between samples. McDonald (1975) gave a concise description of a general approach to testing pattern hypotheses in a single sample, using the method of maximum likelihood, which assumed a Wishart distribution for the observed covariance matrix. This approach, a special case of the analysis of covariance structures, modeled the covariance matrix of the observed variables as

$$\mathbf{\Sigma} = \mathbf{\Delta}\mathbf{P}(\mathbf{\gamma})\mathbf{\Delta} \tag{1}$$

with $\mathbf{P}(\mathbf{\gamma})$ a patterned correlation matrix that is a function of a vector of free parameters $\mathbf{\gamma}$, and $\mathbf{\Delta}$ a diagonal matrix of free scale factors that are, in this application, generally considered nuisance parameters. This model is a special case of all of the commonly employed structural equation models, including the LISREL model. Consequently, maximum likelihood and GLS estimates (and their associated test statistics) may be obtained iteratively using standard methods for nonlinear optimization.

Browne (1977) presented a generalized least squares procedure for testing any correlational pattern hypothesis in a single sample. Browne's development had a significant advantage over the maximum likelihood approach, as the generalized least squares estimators are available in closed form, so nonlinear optimization routines, and their attendant convergence problems, could be avoided with little apparent loss of efficiency or accuracy. Browne (1977) gave an example of how to fit a perfect circumplex model as a correlational pattern hypothesis. Steiger (1980a, 1980b) reviewed

the preceding work, and suggested a modification of Browne's (1977) approach to use the inverse hyperbolic tangent ("Fisher transform") to improve small sample performance. Steiger (1979) implemented his approach in the freeware program *MULTICORR*. McDonald (1975) had shown that OLS estimates are available in closed form when a correlational pattern hypothesis is expressed as a covariance structure model. Browne (1984) showed how to construct an asymptotic chi-square statistic using OLS estimates. The approaches of Browne (1984) and McDonald (1975) may thus be combined to yield a noniterative chi-square test using OLS estimates.

Asymptotically Distribution Free (ADF) Procedures

Although it is seldom mentioned in textbooks, many standard correlational tests are not robust to violations of the assumption of multivariate normality, and are especially sensitive to kurtosis. Layard (1972, 1974) discussed robust tests for comparing covariance matrices and functions of their elements. Browne (1982, 1984) developed general robust methods for the analysis of covariance structures, and coined the term *asymptotically distribution free* (ADF) to describe the procedures. Browne discussed use of ADF analysis of covariance structures to test correlational pattern hypotheses.

Steiger and Hakstian (1982) presented expressions for the asymptotic distribution of correlations under very general distributional assumptions, and suggested that this result be used to modify the GLS approaches, thus yielding robust tests of correlational pattern hypotheses.

Comparing Functions of a Correlation Matrix

The approaches of McDonald (1975), Browne (1977) and Steiger (1980a) had concentrated on tests involving simple Pearson correlations. Olkin and Siotani (1964, 1976) had also discussed the use of the multivariate delta method to perform statistical

pattern hypotheses tests on multiple and partial correlations. Steiger and Browne (1984) developed an alternative approach that allowed comparison of multiple, partial, and/or canonical correlations using a minor modification of GLS and ADF methods for testing simple correlations. Olkin and Press (1995) gave detailed examples of the use of the delta method for comparing correlations of various kinds.

The work discussed above allows tests on single samples, which may be conceptualized as N observations on a vector of p random variables in one population. This includes the case where the same variables are sampled on the same subjects several times. Consequently, this theory can be used to test, say, the hypothesis that a matrix of correlations measured several times on the same individuals has remained stable over time (Steiger, 1980a). However, the papers discussed above do not deal with the case where several *independent samples* (possibly of different size) are taken, and correlations need to be compared across the samples with the full range of pattern hypotheses. For example, although a normal theory test for comparing independent correlation matrices is discussed by Jennrich (1970) and a similar ADF test was developed by Modarres and Jernigan (1993), these tests cannot be computed by Steiger's (1979) *MULTICORR*.

The present paper extends previously available results in several ways:

1. I extend previous theoretical work by Steiger (1980b) in a fairly obvious way to the case of several independent samples, and I provide computer software for performing the analyses.

2. I discuss Steiger's (1980c) observation that unconstrained correlations (those elements of the correlation matrix not constrained to be equal to other correlations or specified numerical values) need not be included in the null hypothesis specification, thereby reducing computational effort in testing some hypotheses. I then prove the surprising result that eliminating these unconstrained correlations has no effect on the chi-

square test statistic, parameter estimates of constrained correlations, or their estimated

standard errors.

3. I demonstrate how the chi-square statistic and parameter estimates may be

obtained without inverting the estimated covariance matrix of correlations *in both the*

normal theory and ADF cases, and whether or not all correlations are included in the

null hypothesis specification.

Pattern Hypothesis Notation

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_A$ be A independent random vectors, of (possibly unequal) order

p_a, $a = 1, 2, \ldots A$, having continuous distributions with mean vectors $\boldsymbol{\mu}_a$, $i = 1, 2, \ldots A$,

and variance-covariance matrices $\boldsymbol{\Sigma}_a$, $a = 1, 2, \ldots A$. Define $\mathbf{P}_a$, $a = 1, 2, \ldots A$, with typical

element $\rho_{ij}^{\langle a \rangle}$, as the population (Pearson product-moment) corrrelation matrix of $\mathbf{x}_a$. Let

$\boldsymbol{\rho}_a$ be the $v_a \times 1$ vectors whose elements are selected lower triangular elements of the $\mathbf{P}_a$,

arranged, for all a, in some consistent fashion. If all non-redundant elements of $\mathbf{P}_a$ are

in $\boldsymbol{\rho}_a$ (and they need not be), then $v_a = (p_a^2 - p_a)/2$. Define V as

$$V = \sum_{a=1}^{A} v_a . \tag{2}$$

A *pattern hypothesis* on the $\mathbf{P}_a$ is a hypothesis that sets of their elements are

equal to each other, or to specified numerical values. Let there be q subsets of the

elements of the $\mathbf{P}_i$ that are hypothesized to (within subset) take on the same unspecified

value γ_i, $i = 1, 2, \ldots q$, and/or possibly w other correlations hypothesized to have a specific

numerical value. A convenient notation for expressing such hypotheses is as follows:

$$H_0 : \rho = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_A \end{bmatrix} = \Delta\gamma + \rho^* \tag{3}$$

where γ is a $q \times 1$ vector of common but unspecified correlations, ρ^* is a $V \times 1$ vector containing w numerically specified values (possibly zero), where appropriate, and zeroes elsewhere, and Δ is a $V \times q$ matrix of zeroes and ones with typical element

$$\delta_{ij} = \partial\rho_i / \partial\gamma_j . \tag{4}$$

Δ is of rank q.

Consider the following simple example. Suppose $A = 2$, $p_1 = p_2 = 3$, and one hypothesizes that $P_1 = P_2$, i.e., that the two 3×3 correlation matrices are equal. The null hypothesis may then be written in the notation of Equations 1 through 3 as

$$\begin{bmatrix} \rho_{21}^{\langle 1 \rangle} \\ \rho_{31}^{\langle 1 \rangle} \\ \rho_{32}^{\langle 1 \rangle} \\ \hline \rho_{21}^{\langle 2 \rangle} \\ \rho_{31}^{\langle 2 \rangle} \\ \rho_{32}^{\langle 2 \rangle} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

Asymptotic Sampling Theory

Following Steiger and Browne (1984), let x_i, x_j, x_k, and x_h be random variables with a multivariate distribution having finite fourth-order moments. Define

$$\mu_i = E(x_i) \tag{6}$$

$$\sigma_{ij} = E(x_i - \mu_i)(x_j - \mu_j) \tag{7}$$

$$\sigma_{ijkh} = E(x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_h - \mu_h) \tag{8}$$

$$\rho_{ij} = \sigma_{ij}\left(\sigma_{ii}\sigma_{jj}\right)^{-1/2} \tag{9}$$

Next consider samples, based on samples of $N = n+1$ independent observations on variates x_i, x_j, x_k, and x_h . We define the sample statistic

$$m_i = N^{-1}\sum_{r=1}^{N} x_{ri} \tag{10}$$

$$s_{ij} = n^{-1}\sum_{r=1}^{N}(x_{ri} - m_i)(x_{rj} - m_j) \tag{11}$$

$$s_{ijkh} = n^{-1}\sum_{r=1}^{N}(x_{ri} - m_i)(x_{rj} - m_j)(x_{rk} - m_k)(x_{rh} - m_h) \tag{12}$$

$$z_{ri} = (x_{ri} - m_i)s_{ii}^{-1/2} \tag{13}$$

$$r_{ij} = s_{ij}\left(s_{ii}s_{jj}\right)^{-1/2} = n^{-1}\sum_{r=1}^{N} z_{ri}z_{rj} \tag{14}$$

$$r_{ijkh} = s_{ijkh}\left(s_{ii}s_{jj}s_{kk}s_{hh}\right)^{-1/2} = n^{-1}\sum_{r=1}^{N} z_{ri}z_{rj}z_{rk}z_{rh} \tag{15}$$

Let $\mathbf{R}_a, a = 1, 2, \dots A$ be A sample correlation matrices, each based on N_a independent observations. Let the vectors $\mathbf{r}_a$, $a = 1, 2, \dots A$ be composed (analogously to the $\boldsymbol{\rho}_a$ in the preceding section) of the lower triangular elements of the $\mathbf{R}_a$. Define $\mathbf{r}_a^*$

as $\mathbf{r}_a^* = n_a^{1/2}\left(\mathbf{r}_a - \boldsymbol{\rho}_a\right)$. It has been established (Hsu, 1949; Isserlis, 1916; Steiger &

Hakstian, 1982) that under very general conditions $\mathbf{r}_a^*$ has an asymptotic distribution that

is $\mathcal{N}\left(\mathbf{0}, \boldsymbol{\Psi}_a\right)$ (i.e., multivariate normal with a null mean vector and covariance matrix

$\boldsymbol{\Psi}_a$). $\boldsymbol{\Psi}_a$ has typical element $\psi_{ij,kh}^{\langle a\rangle} = Cov\left(r_{ij}^{*\langle a\rangle}, \ r_{kh}^{*\langle a\rangle}\right)$ given by

$$
\begin{aligned}
\psi_{ij,kh}^{\langle a\rangle} = {}& \rho_{ijkh}^{\langle a\rangle} + \tfrac{1}{4}\rho_{ij}^{\langle a\rangle}\rho_{kh}^{\langle a\rangle}\left(\rho_{iikk}^{\langle a\rangle} + \rho_{jjkk}^{\langle a\rangle} + \rho_{iihh}^{\langle a\rangle} + \rho_{jjhh}^{\langle a\rangle}\right) \\
& -\tfrac{1}{2}\rho_{ij}^{\langle a\rangle}\left(\rho_{iikh}^{\langle a\rangle} + \rho_{jjkh}^{\langle a\rangle}\right) - \tfrac{1}{2}\rho_{kh}^{\langle a\rangle}\left(\rho_{ijkk}^{\langle a\rangle} + \rho_{ijhh}^{\langle a\rangle}\right)
\end{aligned}
\tag{16}
$$

If the *x*'s have an elliptical distribution with common relative kurtosis coefficient η, then

Equation 16 becomes

$$
\psi_{jk,hm}^{\langle i\rangle} = \tfrac{1}{2}\eta\left\{
\begin{aligned}
&\left(\rho_{jh}^{\langle i\rangle} - \rho_{jk}^{\langle i\rangle}\rho_{kh}^{\langle i\rangle}\right)\left(\rho_{km}^{\langle i\rangle} - \rho_{kh}^{\langle i\rangle}\rho_{hm}^{\langle i\rangle}\right) + \left(\rho_{jm}^{\langle i\rangle} - \rho_{jh}^{\langle i\rangle}\rho_{hm}^{\langle i\rangle}\right)\left(\rho_{kh}^{\langle i\rangle} - \rho_{kj}^{\langle i\rangle}\rho_{jh}^{\langle i\rangle}\right) \\
&+ \left(\rho_{jh}^{\langle i\rangle} - \rho_{jm}^{\langle i\rangle}\rho_{mh}^{\langle i\rangle}\right)\left(\rho_{km}^{\langle i\rangle} - \rho_{kj}^{\langle i\rangle}\rho_{jm}^{\langle i\rangle}\right) + \left(\rho_{jm}^{\langle i\rangle} - \rho_{jk}^{\langle i\rangle}\rho_{km}^{\langle i\rangle}\right)\left(\rho_{kh}^{\langle i\rangle} - \rho_{km}^{\langle i\rangle}\rho_{mh}^{\langle i\rangle}\right)
\end{aligned}
\right\}
\tag{17}
$$

The multivariate normal distribution is a member of the elliptical class with $\eta = 1$.

Substituting this value in the above equation yields the special case formula (Pearson &

Filon, 1898; Hsu, 1949; Olkin & Siotani, 1976) for the covariances of correlations based

on observations from a multivariate normal distribution.

Define $\mathbf{r}^*$ as

$$
\mathbf{r}^* = \mathbf{r} - \boldsymbol{\rho} = \begin{bmatrix} \mathbf{r}_1^* \\ \mathbf{r}_2^* \\ \vdots \\ \mathbf{r}_A^* \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_A \end{bmatrix} - \begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_A \end{bmatrix}
\tag{18}
$$

and $\mathbf{N}$ is a diagonal matrix of the form

$$
\mathbf{N} = \begin{bmatrix} n_1\mathbf{I}_{v_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & n_2\mathbf{I}_{v_2} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & n_A\mathbf{I}_{v_A} \end{bmatrix} \tag{19}
$$

with $\mathbf{I}_{v_a}$ a $v_a \times v_a$ identity matrix.

It follows immediately from well known theory on linear composites that $\mathbf{N}^{\frac{1}{2}}\mathbf{r}^*$ has an asymptotic distribution which is $\mathcal{N}(\mathbf{0}, \mathbf{\Psi})$, where $\mathbf{\Psi}$ is a symmetric block-diagonal covariance matrix with the form

$$
\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Psi}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Psi}_3 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Psi}_A \end{bmatrix} \tag{20}
$$

Estimation Theory and Significance Tests

In practice, estimates of the elements of $\mathbf{\rho}$ under the null hypothesis of Equation 3 are restricted in that estimates of correlations that are hypothesized to be equal are required to be equal in value, and estimates of correlations that are hypothesized to be equal to some numerical value are restricted to be equal to that numerical value. Consequently, $\hat{\mathbf{\rho}}$, the vector of estimates of the elements of $\mathbf{\rho}$ under H_0, may written as

$$
\hat{\mathbf{\rho}} = \Delta\hat{\mathbf{\gamma}} + \mathbf{\rho}^* \tag{21}
$$

and so the problem of estimating $\mathbf{\rho}$ for a given pattern hypothesis can be reduced to finding an estimate $\hat{\mathbf{\gamma}}$ for $\mathbf{\gamma}$.

The "best" estimates are those that minimize an appropriately chosen discrepancy function. The most commonly used discrepancy functions are Ordinary Least Squares (OLS), Generalized Least Squares (GLS), and Maximum Likelihood (ML). I shall deal with OLS and GLS estimators here, since they can be expressed compactly in closed form.

OLS estimators, in the specialized sense I define them here, minimize the sum of squared discrepancies between the vectors $\mathbf{r}_a$ and estimates $\hat{\boldsymbol{\rho}}_a$, where, in the multiple sample case with unequal sample sizes, each squared discrepancy is weighted by n_a. The OLS estimator of $\boldsymbol{\gamma}$ thus minimizes the discrepancy function

$$F_{OLS}(\mathbf{r},\boldsymbol{\gamma})=\left(\mathbf{r}-\Delta\boldsymbol{\gamma}-\boldsymbol{\rho}^*\right)'\mathbf{N}\left(\mathbf{r}-\Delta\boldsymbol{\gamma}-\boldsymbol{\rho}^*\right) \tag{22}$$

where $\mathbf{N}$ is as defined in Equation 19.

$\hat{\boldsymbol{\gamma}}_{OLS}$ is given by

$$\hat{\boldsymbol{\gamma}}_{OLS}=\left(\Delta'\mathbf{N}\Delta\right)^{-1}\Delta'\mathbf{N}\left(\mathbf{r}-\boldsymbol{\rho}^*\right) \tag{23}$$

With $A = 1$, or with equal n_a, $\mathbf{N}$ may be factored out of the above expression, and $\hat{\boldsymbol{\gamma}}_{OLS}$ is equivalent to the ordinary (unweighted) least squares estimator discussed by previous authors, i.e. $\hat{\boldsymbol{\gamma}}_{UOLS}=\left(\Delta'\Delta\right)^{-1}\Delta'\left(\mathbf{r}-\boldsymbol{\rho}^*\right)$. The OLS estimator of $\boldsymbol{\rho}$ is computed from that of $\boldsymbol{\gamma}$ via Equation 21.

Define $\hat{\boldsymbol{\Psi}}$ as a consistent estimator of $\boldsymbol{\Psi}$ under the null hypothesis. We then define generalized least squares estimator $\hat{\boldsymbol{\gamma}}_{GLS}$ as that which minimizes the discrepancy function

$$F_{GLS}(\mathbf{r},\boldsymbol{\gamma})=\left(\mathbf{r}-\Delta\boldsymbol{\gamma}-\boldsymbol{\rho}^*\right)'\hat{\boldsymbol{\Omega}}^{-1}\left(\mathbf{r}-\Delta\boldsymbol{\gamma}-\boldsymbol{\rho}^*\right), \tag{24}$$

where

$$\hat{\mathbf{\Omega}} = \mathbf{N}^{-\frac{1}{2}}\hat{\mathbf{\Psi}}\mathbf{N}^{-\frac{1}{2}} \tag{25}$$

and $\mathbf{N}^{-\frac{1}{2}}$ is the inverse square root matrix of the matrix $\mathbf{N}$ defined in Equation 19.

This estimate is given by

$$\hat{\mathbf{\gamma}}_{GLS} = \left(\mathbf{\Delta}'\hat{\mathbf{\Omega}}^{-1}\mathbf{\Delta}\right)^{-1}\mathbf{\Delta}'\hat{\mathbf{\Omega}}^{-1}\left(\mathbf{r} - \mathbf{\rho}^{*}\right) \tag{26}$$

We now sketch a proof that F_{GLS} has an asymptotic distribution that is χ^{2}_{V-q}, and thus provides a significance test of "badness of fit" of the pattern hypothesis. In what follows, the notation $\mathbf{N} \to \infty$ indicates that $\forall a$, $N_a \to \infty$. First, we recall the following Lemma (Timm, 1975, p. 132) on the distribution of quadratic forms.

Lemma. Let $\mathbf{y}$ be a random vector having a multivariate normal distribution with mean $\mathbf{\mu}$ and covariance matrix $\mathbf{\Sigma}$. Then the quadratic form $\phi = \mathbf{y}'\mathbf{A}\mathbf{y}$ has a chi square distribution with v degrees of freedom and noncentrality parameter $\lambda = \mathbf{\mu}'\mathbf{A}\mathbf{\mu}$ if and only if $\mathbf{A}\mathbf{\Sigma}$ is idempotent and of rank v.

The Lemma will apply to asymptotically multinormal variates if $\text{plim}_{\mathbf{N}\to\infty} \mathbf{A}\mathbf{\Sigma}$ is idempotent and of rank v. To apply the Lemma, we rewrite F_{GLS} as

$$F_{GLS}\left(\mathbf{r},\hat{\mathbf{\gamma}}\right) = \mathbf{y}'\hat{\mathbf{\Psi}}^{-1}\mathbf{y} \tag{27}$$

with

$$\mathbf{y} = \mathbf{N}^{\frac{1}{2}}\left(\mathbf{I} - \mathbf{\Delta}\hat{\mathbf{Q}}\right)\left(\mathbf{r} - \mathbf{\rho}^{*}\right) = \mathbf{N}^{\frac{1}{2}}(\mathbf{r} - \mathbf{\Delta}\hat{\mathbf{\gamma}} - \mathbf{\rho}^{*}) = \mathbf{N}^{\frac{1}{2}}\mathbf{e}, \tag{28}$$

where $\mathbf{e}$ represents the fitted residuals, and

$$\hat{\mathbf{Q}} = \left(\mathbf{\Delta}'\hat{\mathbf{\Omega}}^{-1}\mathbf{\Delta}\right)^{-1}\mathbf{\Delta}'\hat{\mathbf{\Omega}}^{-1} \tag{29}$$

The consistency of $\hat{\Psi}$ implies that

$$\underset{N\to\infty}{\text{plim}}\,\hat{\Psi} = \Psi \tag{30}$$

and hence

$$\underset{N\to\infty}{\text{plim}}\,\hat{Q} = Q = \left(\Delta'\Omega^{-1}\Delta\right)^{-1}\Delta'\Omega^{-1} \tag{31}$$

To apply the Lemma, we need to find the asymptotic distribution of $\mathbf{y}$. From Equations 28 and 31, it follows that $\mathbf{y}$ has the same asymptotic distribution as

$$N^{\frac{1}{2}}\left(\mathbf{I}-\Delta\mathbf{Q}\right)\left(\mathbf{r}-\boldsymbol{\rho}^*\right) = N^{\frac{1}{2}}\mathbf{z} \tag{32}$$

Under a true null hypothesis, $\boldsymbol{\rho}^* = \boldsymbol{\rho}-\Delta\boldsymbol{\gamma}$, and we have

$$\begin{aligned}
\mathbf{z} &= \mathbf{r}-\Delta\mathbf{Q}\left(\mathbf{r}-\boldsymbol{\rho}^*\right)-\boldsymbol{\rho}^* \\
&= \mathbf{r}-\Delta\mathbf{Q}\left(\mathbf{r}-\boldsymbol{\rho}+\Delta\boldsymbol{\gamma}\right)-\boldsymbol{\rho}+\Delta\boldsymbol{\gamma} \\
&= \mathbf{r}-\boldsymbol{\rho}-\Delta\mathbf{Q}(\mathbf{r}-\boldsymbol{\rho})-\Delta\mathbf{Q}\Delta\boldsymbol{\gamma}+\Delta\boldsymbol{\gamma} \\
&= \mathbf{r}-\boldsymbol{\rho}-\Delta\mathbf{Q}(\mathbf{r}-\boldsymbol{\rho})-\Delta\boldsymbol{\gamma}+\Delta\boldsymbol{\gamma} \\
&= \mathbf{r}-\boldsymbol{\rho}-\Delta\mathbf{Q}(\mathbf{r}-\boldsymbol{\rho}) \\
&= (\mathbf{I}-\Delta\mathbf{Q})(\mathbf{r}-\boldsymbol{\rho}).
\end{aligned} \tag{33}$$

Consequently, $\mathbf{y}$ has the same asymptotic distribution as

$$\mathbf{N}^{\frac{1}{2}}\mathbf{z} = \mathbf{N}^{\frac{1}{2}}(\mathbf{I}-\boldsymbol{\Delta}\mathbf{Q})(\mathbf{r}-\boldsymbol{\rho})$$

$$= \mathbf{N}^{\frac{1}{2}}(\mathbf{r}-\boldsymbol{\rho}) - \mathbf{N}^{\frac{1}{2}}\boldsymbol{\Delta}\mathbf{Q}(\mathbf{r}-\boldsymbol{\rho})$$

$$= \mathbf{N}^{\frac{1}{2}}(\mathbf{r}-\boldsymbol{\rho}) - \mathbf{N}^{\frac{1}{2}}\boldsymbol{\Delta}\left(\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}(\mathbf{r}-\boldsymbol{\rho}) \qquad (34)$$

$$= \mathbf{N}^{\frac{1}{2}}(\mathbf{r}-\boldsymbol{\rho}) - \mathbf{N}^{\frac{1}{2}}\boldsymbol{\Delta}\left(\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\mathbf{N}^{\frac{1}{2}}\boldsymbol{\Psi}^{-1}\mathbf{N}^{\frac{1}{2}}(\mathbf{r}-\boldsymbol{\rho})$$

$$= (\mathbf{I}-\mathbf{V})\mathbf{N}^{\frac{1}{2}}(\mathbf{r}-\boldsymbol{\rho}),$$

with

$$\mathbf{V} = \mathbf{N}^{\frac{1}{2}}\boldsymbol{\Delta}\left(\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\mathbf{N}^{\frac{1}{2}}\boldsymbol{\Psi}^{-1}. \qquad (35)$$

From the result of Equation 34, and standard results on expected value and variance of linear composites, it immediately follows that $\mathbf{y}$ has an asymptotic distribution that is $\mathcal{N}\left(\mathbf{0},(\mathbf{I}-\mathbf{V})\boldsymbol{\Psi}(\mathbf{I}-\mathbf{V})'\right)$. To show that F_{GLS} is asymptotically χ^2_{V-q}, we must show that $\boldsymbol{\Psi}^{-1}(\mathbf{I}-\mathbf{V})\boldsymbol{\Psi}(\mathbf{I}-\mathbf{V})'$ is idempotent, and of rank $V-q$. Following some substitution and recombination, we find that

$$\boldsymbol{\Psi}^{-1}(\mathbf{I}-\mathbf{V})\boldsymbol{\Psi}(\mathbf{I}-\mathbf{V})' = (\mathbf{I}-\mathbf{V})' \qquad (36)$$

Idempotency of $(\mathbf{I}-\mathbf{V})'$ is established easily by substitution, and the rank property follows from the rank of $\boldsymbol{\Delta}$, which is q. This completes the proof.

Special cases of Equations 26 through 29 may be obtained by selecting different estimators for $\boldsymbol{\Psi}$ (and hence $\boldsymbol{\Omega}$). Under multivariate normal theory, one may obtain a "single stage GLS" (GLS) estimator by substituting sample correlations for ρ_{ij} in the following equation to obtain estimates of the elements of $\boldsymbol{\Psi}$,

$$\psi^{\langle i \rangle}_{jk,hm} = \left(\rho^{\langle i \rangle}_{jh} - \rho^{\langle i \rangle}_{jk}\rho^{\langle i \rangle}_{kh}\right)\left(\rho^{\langle i \rangle}_{km} - \rho^{\langle i \rangle}_{kh}\rho^{\langle i \rangle}_{hm}\right) + \left(\rho^{\langle i \rangle}_{jm} - \rho^{\langle i \rangle}_{jh}\rho^{\langle i \rangle}_{hm}\right)\left(\rho^{\langle i \rangle}_{kh} - \rho^{\langle i \rangle}_{kj}\rho^{\langle i \rangle}_{jh}\right)$$

$$+ \left(\rho^{\langle i \rangle}_{jh} - \rho^{\langle i \rangle}_{jm}\rho^{\langle i \rangle}_{mh}\right)\left(\rho^{\langle i \rangle}_{km} - \rho^{\langle i \rangle}_{kj}\rho^{\langle i \rangle}_{jm}\right) + \left(\rho^{\langle i \rangle}_{jm} - \rho^{\langle i \rangle}_{jk}\rho^{\langle i \rangle}_{km}\right)\left(\rho^{\langle i \rangle}_{kh} - \rho^{\langle i \rangle}_{km}\rho^{\langle i \rangle}_{mh}\right) \qquad (37)$$

then using the resulting $\hat{\boldsymbol{\Psi}}$ in Equations 26 through 29. However, the null hypothesis may be incorporated into the estimates of the elements of $\boldsymbol{\Psi}$ by using OLS estimates for the ρ_{ij} instead of sample correlations in Equation 37. The resulting estimates are referred to as "2-stage GLS" (TSGLS) estimators. In a similar vein, one may compute "single-stage ADF" (ADF) estimators by constructing $\hat{\boldsymbol{\Psi}}$ with sample correlations and standardized 4th order moments in Equation 16, and "2-stage ADF" (TSADF) estimators by using OLS estimates instead of sample correlations. Steiger and Hakstian (1982) give an example of calculation of a test statistic for comparing two dependent correlations using the TSADF approach.

Define **e** as the fitted residuals, i.e.

$$\mathbf{e} = \mathbf{r} - \Delta\hat{\boldsymbol{\gamma}} - \boldsymbol{\rho}^{*} \tag{38}$$

Estimates of the covariance matrix of the fitted residuals may be calculated as

$$\widehat{\mathrm{Var}}(\mathbf{e}) = \mathbf{N}^{-1/2}\left(\hat{\boldsymbol{\Omega}} - \Delta\left(\Delta'\hat{\boldsymbol{\Omega}}^{-1}\Delta\right)^{-1}\Delta' \right)\mathbf{N}^{-1/2} \tag{39}$$

Asymptotically Distribution Free Procedures

The generalized least squares testing procedures described in the preceding sections, in particular the test statistic of Equation 24, holds so long as **r** is an asymptotically unbiased and multinormal estimate of $\boldsymbol{\rho}$, and the estimated variance-covariance matrix (i.e., $\hat{\boldsymbol{\Omega}}_{OLS}$ in Equation 24) for the is consistent. If the population distribution departs from multivariate normality, **r** will, under very general conditions, remain asymptotically normal but the matrix $\boldsymbol{\Psi}$ with elements defined as in Equation 37 will no longer be correct. As Steiger and Hakstian (1982) have pointed out, this problem

can be alleviated by substituting, in place of Equation 37, a more general formula, Equation 16, which holds whether or not the population distribution is multivariate normal. This formula, which yields the *asymptotically distribution free* (ADF) correlational procedures, should be employed with caution, as it requires sample estimates of 4th order moments. These estimates have large standard errors at small to moderate sample sizes, so convergence of the ADF test statistic to its asymptotic distribution is often considerably slower than the normal theory variant. Employing the ADF formula when it is not needed (i.e., when the population distribution *is* multivariate normal) may result in a considerable loss of power.

It should be noted that, with $k > 1$, $\mathbf{\Delta}$ of the form in Equation 40,

$$\mathbf{\Delta} = \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix} \tag{40}$$

and TSGLS estimates, a slight modification of the statistic F_{GLS} (produced by using N_a rather than $n_A = N_A - 1$ in the matrix $\mathbf{N}$) is equivalent to the one given by Jennrich (1970) for testing the equality of 2 or more correlation matrices. If all elements of $\mathbf{P}$ (including those that are not constrained by the null hypothesis), are included in the hypothesis vector, then F_{GLS} is formally equivalent to the statistic developed by Browne (1977) (although the computational formulas differ, and the relative computational efficiency depends on a particular application).

Computational Considerations

Interpreted literally, the equations for the chi-square statistics discussed above would appear to require, for larger problems, very large amounts of computational space.

The difficulties stem from the fact that as the order of a correlation matrix increases, the number of non-redundant elements grows very rapidly. For example, a 20×20 correlation matrix has 190 non-redundant correlations. If $A = 3$, and all correlations are involved in the null hypothesis, the matrix $\mathbf{\Psi}$ would then be a block diagonal matrix of order 570×570. Each of the three 190×190 blocks would contain 17,955 non-redundant elements. Inverting such a matrix involves significant computational effort. In view of such facts, it is important to consider methods for eliminating unnecessary computations.

Discussion of all nontrivial aspects of computational optimization for pattern hypothesis tests is beyond the scope of this article. Below I consider aspects of optimization that are related primarily to the statistical theory discussed in this article, rather than the niceties of programming in any specific computer language. With respect to the latter, let us simply say that (1) some model matrices are sparse, and can be stored very efficiently (as we illustrate below in the case of $\mathbf{\Delta}$), and (2) computational efficiency can be improved in many places by careful employment of specialized routines for storing and manipulating symmetric matrices.

We will now discuss theoretical developments that are particularly relevant to computational efficiency. First, we point out a distinction between the theory for GLS procedures as presented here, and in Steiger (1980b. 1980c), and that given by Browne (1977). In Browne's (1977) derivation of single sample procedures, the vector $\mathbf{\rho}$ is assumed to hold all $\left(p^2 + p\right)/2$ elements of $\mathbf{P}$. The theory presented in this article, and in Steiger (1980b, 1980c) requires only the *restricted* elements of $\mathbf{P}$ to be included in $\mathbf{\rho}$, although unrestricted elements may be included. For some types of hypotheses, this difference can lead to substantial differences in computational effort. On the other hand, Browne (1977) contained computational simplifications that can yield greater efficiency

than the methods discussed here when many of the elements of the correlation matrix are constrained.

Jennrich (1970) presented methods that allowed the computation of $\hat{\mathbf{\Omega}}^{-1}$ without directly inverting $\hat{\mathbf{\Omega}}$, but instead by inverting two $p \times p$ matrices. This method can produce substantial gains in computing time. The method, however, assumed that all elements of the correlation matrix being tested are included in $\boldsymbol{\rho}$, and, moreover, only handled the normal theory case, so that they were not applicable to the method discussed in the preceding paragraph. In what follows, I present an alternative approach that allows computation of $\hat{\mathbf{\Omega}}^{-1}$ without directly inverting $\hat{\mathbf{\Omega}}$, but which can be applied either in the normal theory or ADF case, and whether or not all correlations are included in $\boldsymbol{\rho}$.

In the earlier development in this article, I assumed for clarity of exposition that the correlations included in the null hypothesis specification were sorted by group, i.e., all elements from group 1 were first, followed by elements from group 2, etc. However, this assumption is actually not necessary. Two simple adjustments in the notation are needed: (a) the matrix $\mathbf{N}$ in Equation 19 is redefined to be a diagonal matrix with each diagonal element the sample size N (instead of $n = N - 1$) on which the corresponding correlation is based, and (b) the matrix $\mathbf{\Psi}$ has elements that are zero if the correlations come from different samples (since they are then independent), and otherwise are as defined previously. Formulas for the estimators and test statistics are otherwise unchanged.

I now demonstrate how GLS estimators, and the associated chi-square statistic, may be obtained without inverting the estimated covariance matrix of correlations (or transformed correlations.) Rather than discuss equations specific to the three chi-square statistics we have discussed, we will present these results in terms of a general computational form applicable, with minimal effort, to all three cases. To develop the results, we will need the following

Lemma (Khatri, 1966). Suppose that $\mathbf{S}$ is a $p \times p$ positive definite matrix. If $\mathbf{A}$ (of order $p \times m$) and $\mathbf{B}$ (of order $p \times (p-m)$) are of ranks m and $(p-m)$, respectively, and if

$$\mathbf{B}'\mathbf{A} = \mathbf{0} \tag{41}$$

then

$$\mathbf{B}\left(\mathbf{B}'\mathbf{S}\mathbf{B}\right)^{-1}\mathbf{B}' = \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{S}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{S}^{-1} \tag{42}$$

Recall that expressions for GLS estimators are of the form

$$\hat{\boldsymbol{\gamma}} = \left(\boldsymbol{\Delta}'\mathbf{U}^{-1}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\mathbf{U}^{-1}\left(\mathbf{r} - \boldsymbol{\rho}^{*}\right) \tag{43}$$

This can be re-written as

$$\hat{\boldsymbol{\gamma}} = \left(\boldsymbol{\Delta}'\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\left\{\boldsymbol{\Delta}\left(\boldsymbol{\Delta}'\mathbf{U}^{-1}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\right\}\mathbf{U}^{-1}\left(\mathbf{r} - \boldsymbol{\rho}^{*}\right) \tag{44}$$

which, upon application of Khatri's (1966) Lemma, becomes

$$\begin{aligned}
\hat{\boldsymbol{\gamma}} &= \left(\boldsymbol{\Delta}'\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\left\{\mathbf{U} - \mathbf{U}\boldsymbol{\Phi}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}'\mathbf{U}\right\}\mathbf{U}^{-1}\left(\mathbf{r} - \boldsymbol{\rho}^{*}\right) \\
&= \left(\boldsymbol{\Delta}'\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\left\{\left(\mathbf{r} - \boldsymbol{\rho}^{*}\right) - \mathbf{z}\right\}
\end{aligned} \tag{45}$$

where

$$\mathbf{z} = \mathbf{U}\boldsymbol{\Phi}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}'\left(\mathbf{r} - \boldsymbol{\rho}^{*}\right) \tag{46}$$

and $\boldsymbol{\Phi}$ is a matrix which satisfies Equation 34 with respect to $\boldsymbol{\Delta}$, i.e., if $\boldsymbol{\Delta}$ is of order $V \times q$, then $\boldsymbol{\Phi}$ is of order $V \times (V - q)$, and

$$\mathbf{\Phi}'\mathbf{\Delta} = \mathbf{0} \tag{47}$$

Equation 45 shows that GLS estimators can be computed without inverting the matrix $\mathbf{U}$. It also shows that GLS estimators may be thought of as OLS estimators based on an "adjusted" correlation matrix.

An efficient computational approach is as follows:

1. Construct the matrix $\mathbf{\Phi}$.

2. Compute the matrix $\mathbf{\Phi}'\mathbf{U}\mathbf{\Phi}$. To avoid unnecessary multiplication by zeroes in the A-sample case, take advantage of the block diagonal structure of $\mathbf{U}$ by row-partitioning $\mathbf{\Phi}$ into A submatrices of order $v_i \times (V - q)$. Then use the equality

$$\mathbf{\Phi}'\mathbf{U}\mathbf{\Phi} = \sum_{i=1}^{A} \mathbf{\Phi}_i' \mathbf{U}_i \mathbf{\Phi}_i \tag{48}$$

3. Solve the linear equation system

$$(\mathbf{\Phi}'\mathbf{U}\mathbf{\Phi})\mathbf{y} = \mathbf{\Phi}'\mathbf{r} = \mathbf{b} \tag{49}$$

for $\mathbf{y}.$

Then compute $\mathbf{z}$ in Equation 46 as $\mathbf{z} = \mathbf{U}\mathbf{\Phi}\mathbf{y}$.

4. Compute the vector $\mathbf{r} - \boldsymbol{\rho}^* - \mathbf{z}$.

5. Compute $\hat{\boldsymbol{\gamma}}$ from Equation 45, keeping in mind that neither the matrix $\mathbf{\Delta}$, nor, for that matter $(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'$, need be computed or stored in matrix form, since $\mathbf{\Delta}$ has at most one nonzero element in each row, and that nonzero element is always 1. An efficient internal representation of $\mathbf{\Delta}$ could be an integer vector containing the column index of the 1 in each row of $\mathbf{\Delta}$ (or a zero if the row is null).

In the course of computing $\hat{\boldsymbol{\gamma}}$, we obtain the quantities $\mathbf{y}$ and $\mathbf{b}$. From these, the χ^2_{V-q} test statistic may be obtained using the computational form

$$
\begin{aligned}
\hat{F} &= \left(\mathbf{r}-\boldsymbol{\rho}^*-\Delta\hat{\boldsymbol{\gamma}}\right)'\mathbf{U}^{-1}\left(\mathbf{r}-\boldsymbol{\rho}^*-\Delta\hat{\boldsymbol{\gamma}}\right) \\
&= \left(\mathbf{r}-\boldsymbol{\rho}^*\right)'\left\{\mathbf{I}-\mathbf{U}^{-1}\Delta\left(\Delta'\mathbf{U}^{-1}\Delta\right)^{-1}\Delta'\right\}\mathbf{U}^{-1}\left\{\mathbf{I}-\Delta\left(\Delta'\mathbf{U}^{-1}\Delta\right)^{-1}\Delta'\mathbf{U}^{-1}\right\}\left(\mathbf{r}-\boldsymbol{\rho}^*\right) \\
&= \left(\mathbf{r}-\boldsymbol{\rho}^*\right)'\left\{\mathbf{U}^{-1}-\mathbf{U}^{-1}\Delta\left(\Delta'\mathbf{U}^{-1}\Delta\right)^{-1}\Delta'\mathbf{U}^{-1}\right\}\left(\mathbf{r}-\boldsymbol{\rho}^*\right) \qquad (50)\\
&= \left(\mathbf{r}-\boldsymbol{\rho}^*\right)'\boldsymbol{\Phi}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}'\left(\mathbf{r}-\boldsymbol{\rho}^*\right) \\
&= \mathbf{b}'\mathbf{y}
\end{aligned}
$$

A consistent estimate of the asymptotic variances and covariances of the parameter estimates can be obtained without inverting $\mathbf{U}$ using the following result:

$$
\begin{aligned}
\boldsymbol{\Theta} = \widehat{\mathrm{Cov}}\left(\hat{\boldsymbol{\gamma}},\,\hat{\boldsymbol{\gamma}}\right) &= \left(\Delta'\Delta\right)^{-1}\Delta'\left\{\Delta\left(\Delta'\mathbf{U}^{-1}\Delta\right)^{-1}\Delta'\right\}\left(\Delta'\Delta\right)^{-1}\Delta' \\
&= \left(\Delta'\Delta\right)^{-1}\Delta'\left\{\mathbf{U}-\mathbf{U}\boldsymbol{\Phi}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}'\mathbf{U}\right\}\left(\Delta'\Delta\right)^{-1}\Delta' \qquad (51)\\
&= \mathbf{G}'\mathbf{U}\mathbf{G}-\mathbf{H}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\mathbf{H}',
\end{aligned}
$$

where $\mathbf{G}=\Delta\left(\Delta'\Delta\right)^{-1}$ and $\mathbf{H}=\mathbf{G}'\mathbf{U}\boldsymbol{\Phi}$. Estimated standard errors for the estimates in $\hat{\boldsymbol{\gamma}}$ can be obtained by taking the square root of corresponding diagonal elements of $\boldsymbol{\Theta}$.

We next use the preceding results to demonstrate that elimination of the unconstrained correlations from the null hypothesis specification has no effect on (a) the chi-square test statistic, or (b) the estimates of the constrained correlations. First, we need the following

Lemma. If

$$
\Delta = \begin{bmatrix} \Delta_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad (52)
$$

and if

$$\Phi'\Delta = 0 \tag{53}$$

then

$$\Phi = \begin{bmatrix} \Phi_1 \\ 0 \end{bmatrix} \tag{54}$$

where $\Phi_1'\Delta = 0$.

Proof. Define

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \tag{55}$$

Then $\Phi'\Delta = \begin{bmatrix} \Phi_1'\Delta_1 & \Phi_2' \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and hence $\Phi_1'\Delta = 0$ and $\Phi_2 = 0$ □

Suppose we consider a case where some of the correlations in the null hypothesis are *not* constrained to be equal to any other correlation or to specific numerical values. We partition ρ as

$$\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{56}$$

 so that all constrained correlations are in ρ_1, and remaining correlations are in ρ_2. In that case, since each unconstrained correlation will be equal to a unique element of γ, Δ will be of the form

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & I \end{bmatrix} \tag{57}$$

and the null hypothesis may be partitioned as

$$\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \Delta\gamma + \rho^* = \begin{bmatrix} \Delta_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} \rho_1^* \\ 0 \end{bmatrix} \tag{58}$$

If **U** (in Equations 43–51) is partitioned to correspond with our partition of ρ, i.e.,

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{bmatrix},$$

and one applies the facts (easily established by substitution) that

$$(\Delta'\Delta)^{-1} = \begin{bmatrix} \left(\Delta_1'\Delta_1\right)^{-1} & 0 \\ 0 & I \end{bmatrix}, \tag{59}$$

$$\Phi'(\mathbf{r} - \rho^*) = \Phi_1'(\mathbf{r}_1 - \rho_1^*), \tag{60}$$

and

$$\Phi'\mathbf{U}\Phi = \Phi_1'\mathbf{U}_{11}\Phi_1 \tag{61}$$

then simple substitution and recombination, via Equation 45, establishes that

$$\hat{\gamma}_1 = \left(\Delta_1'\mathbf{U}_{11}^{-1}\Delta_1\right)^{-1}\Delta_1'\mathbf{U}_{11}^{-1}(\mathbf{r}_1 - \rho_1^*) \tag{62}$$

and that the chi-square test statistic is

$$\hat{F} = \left(\mathbf{r}_1 - \rho_1^* - \Delta_1\hat{\gamma}_1\right)'\mathbf{U}_{11}^{-1}\left(\mathbf{r}_1 - \rho_1^* - \Delta_1\hat{\gamma}_1\right) \tag{63}$$

For example, to prove Equation 62, we combine Equations 45, 46, and 59, obtaining

$$\hat{\gamma} = \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} = (\Delta'\Delta)^{-1}\Delta'\left\{(\mathbf{r}-\boldsymbol{\rho}^*)-\mathbf{z}\right\}$$

$$= \begin{bmatrix} (\Delta_1'\Delta_1)^{-1}\Delta_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} (\mathbf{r}_1-\boldsymbol{\rho}_1^*)-\mathbf{z}_1 \\ \mathbf{r}_2-\mathbf{z}_2 \end{bmatrix} \qquad (64)$$

$$= \begin{bmatrix} (\Delta_1'\Delta_1)^{-1}\Delta_1'\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*-\mathbf{z}_1\right) \\ \mathbf{r}_2-\mathbf{z}_2 \end{bmatrix}$$

with

$$\begin{aligned}
\mathbf{z}_1 &= \mathbf{U}_{11}\boldsymbol{\Phi}_1\left(\boldsymbol{\Phi}_1'\mathbf{U}_{11}^{-1}\boldsymbol{\Phi}_1\right)^{-1}\boldsymbol{\Phi}_1'\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right) \\
&= \mathbf{U}_{11}\left\{\mathbf{U}_{11}^{-1}-\mathbf{U}_{11}^{-1}\Delta_1\left(\Delta_1'\mathbf{U}_{11}^{-1}\Delta_1\right)^{-1}\Delta_1'\mathbf{U}_{11}^{-1}\right\}\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right) \qquad (65) \\
&= \left\{\mathbf{I}-\Delta_1\left(\Delta_1'\mathbf{U}_{11}^{-1}\Delta_1\right)^{-1}\Delta_1'\mathbf{U}_{11}^{-1}\right\}\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{z}_2 &= \mathbf{U}_{21}\boldsymbol{\Phi}_1\left(\boldsymbol{\Phi}_1'\mathbf{U}_{11}\boldsymbol{\Phi}_1\right)^{-1}\boldsymbol{\Phi}_1'\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right) \\
&= \mathbf{U}_{21}\mathbf{U}_{11}^{-1}\left\{\mathbf{I}-\Delta_1\left(\Delta_1'\mathbf{U}_{11}^{-1}\Delta_1\right)^{-1}\Delta_1'\mathbf{U}_{11}^{-1}\right\}\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right)
\end{aligned} \qquad (66)$$

Combining Equations 64–66, we find

$$\begin{aligned}
\hat{\gamma}_2 &= \mathbf{r}_2-\mathbf{z}_2 = \mathbf{r}_2-\mathbf{U}_{21}\mathbf{U}_{11}^{-1}\left(\mathbf{r}_1-\Delta_1\hat{\gamma}_1-\boldsymbol{\rho}_1^*\right) \\
&= \mathbf{r}_2-\mathbf{U}_{21}\mathbf{U}_{11}^{-1}\hat{\mathbf{e}}_1
\end{aligned} \qquad (67)$$

where $\hat{\mathbf{e}}_1$ is the vector of residual correlations, i.e., the difference between the estimates

and the sample correlations.

The upper partition of Equation 64, when compared to Equation 45, demonstrates

that adding unconstrained correlations to the hypothesis specification does not affect the

GLS estimates of the constrained correlations in $\hat{\boldsymbol{\gamma}}_1$. It also demonstrates that, in a single sample, GLS estimates may be thought of as OLS estimates computed on an "adjusted" set of sample correlations. When the adjustment is zero, GLS and OLS estimates are identical.

The fact that the chi-square statistic is unchanged by the addition of unconstrained correlations may be proven in a similar manner. From Equations 50, 60, and 61 we have

$$
\begin{aligned}
\hat{F} &= \left(\mathbf{r}-\boldsymbol{\rho}^*\right)' \boldsymbol{\Phi}\left(\boldsymbol{\Phi}'\mathbf{U}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}'\left(\mathbf{r}-\boldsymbol{\rho}^*\right) \\
&= \left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right)' \boldsymbol{\Phi}_1\left(\boldsymbol{\Phi}_1'\mathbf{U}_{11}^{-1}\boldsymbol{\Phi}_1\right)^{-1}\boldsymbol{\Phi}_1'\left(\mathbf{r}_1-\boldsymbol{\rho}_1^*\right)
\end{aligned}
\tag{68}
$$

Comparing this to the penultimate line of Equation 50 establishes that the chi-square statistic calculated with unconstrained correlations included in the hypothesis statement is equivalent to the statistic without the correlations included.

Software for Performing Pattern Hypothesis Tests

There are a number of computer software programs *specifically* designed to test correlational pattern hypotheses. Some key points of differentiation are (1) whether the software can perform ADF as well as normal theory tests, and (2) whether the software can perform tests on more than one sample.

MULTICORR (Steiger, 1979) is a FORTRAN program for testing any pattern hypothesis on a single sample. MULTICORR is freeware, and employs the Fisher transform, as discussed in Steiger (1980a, 1980b). The program is available for download from the website http://www.interchange.ubc.ca/steiger/homepage.htm, or by email from the author. MULTICORR has some serious limitations, in that it is currently designed to operate on single samples, and is limited to correlation matrices no larger than 18×18.

FITCORS (Mels, 2000) is a general program for fitting correlational structures of the form

$$H_0 : \boldsymbol{\rho} = \mathbf{m}(\boldsymbol{\theta}) \tag{69}$$

where $\boldsymbol{\theta}$ is a vector of free parameters, and $\mathbf{m}$ is a vector model function. Clearly, Equation 69 includes Equation 3 as a special case, so FITCORS can be used to fit correlational models that are not pattern hypothesis tests. However, for models that are neither path models nor correlational pattern hypotheses, the user is required to write and compile a function implementing the model function $\mathbf{m}$. FITCORS implements a variety of estimation methods, including TSGLS, GLS, ADF, and TSADF chi-square statistics. The program runs under Windows, and is available from the author, Gerhard Mels, whose email address at this writing is mels@ssicentral.com. The program currently performs general tests on one sample only.

WBCORR(Within-Between CORRlational Tests) for Mathematica is a freeware *Mathematica* package for performing correlational pattern hypothesis tests in one or more samples with possibly unequal sample sizes. The program, which requires the well-known symbolic algebra program *Mathematica*, includes options for analyzing raw data or correlation matrices, and implements TSGLS, GLS, ADF, and TSADF chi-square statistics. WBCORR is available from the first author by email, or may be downloaded from his website.

Other approaches to performing correlational pattern hypotheses are available besides those discussed above. For example, correlational pattern hypothesis tests may be performed with any common covariance structure analysis software program by implementing the method discussed by McDonald (1975). Although these programs are designed primarily to test hypotheses on a covariance matrix, they can be employed to

test correlational pattern hypotheses. In particular, my own program SEPATH (part of *Statistica*) can perform correlational pattern hypotheses directly on one or more samples, by simply selecting "Analyze Correlations" as a program option, and Michael Browne's RAMONA (part of *Systat*) offers similar capabilities for a single sample. Correlation-based methods, i.e., those based on the sampling theory of a correlation matrix, have somewhat different performance characteristics than covariance-based procedures. For a comparison of the relative performance of covariance-based and correlation-based methods for analyzing correlational pattern hypothesis, see Monte Carlo experiments by Steiger (1980b), Fouladi(1996, 2000), and Mels(2000).

Some Numerical Examples

The following examples illustrate some of the procedures that can be performed using the theory in this paper. All these tests can be performed using WBCORR. After loading in the package to *Mathematica* with the command "<<wbcorr.m", the user simply inputs data, specifies a hypothesis, and invokes a single command.

Single Sample Tests

In this section, we examine hypothesis tests that involve only a single group. The first 3 examples will be based on the raw data shown in Table 1. These data are 25 pseudorandom samples of 6 independent variates all having a lognormal (0,1) distribution. The data were rounded at two decimals, and the numbers actually processed are exactly as shown in the table. Assume in these examples that the 6 variables are all observed on the same group of subjects.

Equicorrelation Hypothesis. Suppose that you hypothesize that the first 3 variables have an *equicorrelation* structure. This structural hypothesis assumes that all

off-diagonal elements of the correlation matrix are equal, so the null hypothesis, in the

notation of Equation 3, is represented as follows (superscripts being unnecessary when

there is only one sample.)

$$
\begin{bmatrix} \rho_{21}^{\langle 1 \rangle} \\ \rho_{31}^{\langle 1 \rangle} \\ \rho_{32}^{\langle 1 \rangle} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [\gamma_1] + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_1 \\ \gamma_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{70}
$$

To test this hypothesis with WBCORR, one codes the left and right sides of the equation

into a single hypothesis matrix of the form

$$
\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 & 0 \end{bmatrix}
\tag{71}
$$

Each row of the hypothesis matrix represents a statement about a correlation that is

constrained by the null hypothesis. The statement is of the form

$$
\text{group, row, column, parameter tag, fixed value} \tag{72}
$$

If the correlation is fixed at a specific numerical value, the fixed value is entered as the

ρ^* value, (and the parameter tag is 0). If the parameter tag is nonzero, then the fixed

value is not used, and is generally entered as zero.

If the raw data are read into *Mathematica* and assigned to a matrix called "data",

and the above hypothesis matrix is assigned to a matrix called

"equicorrelationHypothesis," testing the model requires entry of the single command

"ComputeChiSquare[{data},equicorrelationHypothesis, {25}, DataType->RawData]".

WBCORR responds by producing output that describes the results of hypothesis testing.

The typical *Mathematica* session performing computation for the above problem is shown

in Appendix 1. Scanning the output, we see a number of items that are common to such

analyses. After printing the input correlation matrix, WBCORR generates OLS estimates

of the entire input correlation matrix. Note that the estimates for ρ_{21}, ρ_{31}, and ρ_{32} are all

equal to a common value, i.e., .0642778. The next output section includes a list of

parameter estimates for the constrained free parameters. In this case, three correlations are

constrained to be equal, and have a parameter tag of 1. The two-stage GLS estimate for

the parameter is identical to the OLS estimate. The chi-square test for the null hypothesis,

reported under "Significance Test Results," is *not* rejected, and so remains tenable. How

that the tests for multivariate normality based on the Mardia (1970) indices of

multivariate skewness and kurtosis are both statistically significant, indicating that the

default test (which assumes multivariate normality) may be inaccurate. In such a

situation, it is advisable to compute an ADF version of the test statistic. Re-analysis with

the two-stage ADF test statistic yields the results shown in Part b of Appendix 1. Three

important changes can be observed when comparing the two-stage ADF and two-stage

GLS results. First, the parameter estimate for the common correlation has changes, and

so has its standard error. Second, the value of the chi-square statistic and its associated

probability level have changed. In this case, the statistical decision remains the same, i.e.,

not to reject the null hypothesis. However, in many cases shifting to an ADF statistic can

change the outcome of a significance test. Third, the parameter estimate for parameter 1

is no longer equal to the corresponding OLS estimate.

One can also employ a range of estimation methods, including two-stage GLS and

two-stage ADF, to test the equicorrelation matrix with the freeware program FITCORS

(Mels, 2000). FITCORS, however, employs a somewhat different command language that

represents the correlation matrix as a path diagram. For example, the equicorrelation

hypothesis is represented as follows:

```
MANIFEST
X1 X2 X3
MODEL
X1 <--> (X1, 0, 1.0) (X2, 1, *) (X3, 1, *)  /
X2 <--> (X2, 0, 1.0) (X3, 1, *) ;
END OF MODEL
```

The code in the first line after "MODEL" establishes a fixed value of 1.0 for the

correlation between X1 and itself, and a free parameter with an arbitrary starting value for

the correlation between X1 and X2, and the correlation between X1 and X3. The code in

the second line of the model specification establishes a fixed value of 1.0 for the

correlation between X2 and itself, and a free parameter with a tag of 1 for the correlation

between X1 and X3.

Equality of Two Dependent Correlations. A common pattern hypothesis is to test

whether two predictor variables predict a criterion equally well. This is a hypothesis of

the form

$$\rho_{21} = \rho_{32} = \rho_1 \tag{73}$$

To test such a hypothesis with WBCORR, one uses the following hypothesis matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 & 0 \end{bmatrix} \tag{74}$$

Mathematica input and output for such a hypothesis test using the two-stage ADF

estimation method are shown in Appendix 2.

Perfect Circumplex (Symmetric Circulant) Structure. A perfect circumplex, or

symmetric circulant hypothesis (Wiggins, Steiger, & Gaelick, 1981) implies that

correlations are exactly equal in diagonal strips, and is a special case of the circumplex

model discussed by Browne (1992). For 6 variables, this model states that the correlation matrix is of the form

$$P = \begin{bmatrix} 1 & & & & & \\ \rho_1 & 1 & & & & \\ \rho_2 & \rho_1 & 1 & & & \\ \rho_3 & \rho_2 & \rho_1 & 1 & & \\ \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 & \\ \rho_1 & \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix} \tag{75}$$

The WBCORR hypothesis matrix, and output for this model are shown in Appendix 3. For replicability, we use a random number seed (i.e., 123). Note that, in this case, both tests for multivariate normality have p-values close to .50, indicating that there is no evidence to reject multivariate normality. Using the two-stage GLS estimation method, we obtain a chi-square test for the circumplex hypothesis that is not significant, indicating that the hypothesis of a perfect circumplex remains tenable for these data.

FITCORS can also test this hypothesis, using the following model statements:

```
X1  <--> (X1, 0, 1.0) (X2, 1, *) (X3, 2, *) (X4, 3, *) /
X1  <--> (X5, 4, *) (X6, 5, *) /
X2  <--> (X2, 0, 1.0) (X3, 1, *) (X4, 2, *) (X5, 3, *) /
X2  <--> (X6, 4, *) /
X3  <--> (X3, 0, 1.0) (X4, 1, *) (X5, 2, *) (X6, 3, *) /
X4  <--> (X4, 0, 1.0) (X5, 1, *) (X6, 2, *) /
X5  <--> (X5, 0, 1.0) (X6, 1, *) /
X6  <--> (X6, 0, 1.0) ;
```

In this example, we have tested the very highly constrained hypothesis of perfect circumplex structure. More general circumplex models that are often of significant interest in practical applications are discussed in detail by Browne (1992). These models require software that can test structural models that are more general than a pattern hypothesis. A freeware computer program (CIRCUM) for fitting these more general models may be downloaded from the author's website.

Equality of Correlation Matrices Over Time. If a set of k variables is observed m times on N individuals, the resulting correlation matrix may be viewed as N observations on km dependent variables. Equality of correlation matrices over time may then be tested by constraining the correlations across the m occasions to be equal to each other. Consider, for example, the correlations among 3 variables measured on the same individuals on two occasions. In this case, X_1, X_2, and X_3 represent the three variables on the first occasion, while X_4 through X_6 represent the same 3 variables on the second measurement occasion. Consequently, the hypothesis that the correlations among the 3 variables have not changed over time is equivalent to the following:

$$H_0 : \rho_{21} = \rho_{54} = \gamma_1, \ \rho_{31} = \rho_{64} = \gamma_2, \ \rho_{32} = \rho_{65} = \gamma_3 \tag{76}$$

In WBCORR, we express such a hypothesis with the following hypothesis matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 1 & 3 & 2 & 3 & 0 \\ 1 & 5 & 4 & 1 & 0 \\ 1 & 6 & 4 & 2 & 0 \\ 1 & 6 & 5 & 3 & 0 \end{bmatrix}$$

A sample correlation matrix representing measurements on 3 variables on two separate occasions will be of order 6×6 correlation matrix, and may be analyzed with WBCORR, MULTICORR, or FITCORS to test the hypothesis of equality of correlation matrices over time. For brevity, we do not show computer output and input files here (although they are distributed with the WBCORR software).

Multiple Sample Tests

In this section, we examine hypothesis tests performed on two or more correlation matrices sampled from independent groups. For simplicity, we shall confine our numerical examples to two correlation matrices, since the extension to more than two groups is straightforward.

Equality of Independent Correlation Matrices. Table 3 shows two correlation matrices, representing correlations between GPA and SAT-Verbal and SAT-Math scores for two groups of unequal size. To test whether the two correlation matrices are equal, we employ the following hypothesis matrix with WBCORR:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 1 & 3 & 2 & 3 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 2 & 3 & 0 \end{bmatrix}$$

The first three lines of the hypothesis matrix simply establish that the elements of the correlation matrix for the first group are all free parameters, with parameter tags 1,2, and 3. The next three lines constrain the elements of the correlation matrix in the second group to be equal to those in the first group. WBCORR input and output for the example are shown in Appendix 4.

Equality of Predictor-Criterion Correlations in Two Groups. Suppose that, rather than comparing all the correlations for the data in Table 3, we wished to compare only the predictor-criterion correlations. For example, we might wish to test whether the correlations between the predictors (SAT-Verbal and SAT-Math) and the criterion (GPA) differed in the two groups. In this case, the hypothesis matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 & 0 \end{bmatrix}$$

The TSGLS chi-square statistic is 7.57 with 2 degrees of freedom (p = .022721), and is significant at the .05 level. For brevity, we do not show computer output and input files here (although they are distributed with the WBCORR software).

Equality of Circumplex Structures in Two Groups. One can test the simultaneous hypothesis that two groups have the same correlation matrix *and* that the correlation matrix is a perfect circumplex. In Appendix 5, we present Input and Output for a demonstration of such calculations, using WBCORR. In the first part of the demonstration, we use the random number generation capabilities of *Mathematica* to produce two sets of data, with different sample sizes, both of which fit the same circumplex structure perfectly. In the second part of the demonstration, we test the hypothesis that the two groups have identical circumplex structure, i.e., have correlation matrices that are the same *and* that fit a circumplex perfectly. Not surprisingly, the chi-square statistic is not significant.

Conclusions

In this paper, I have presented general theory for multiple-sample hypothesis tests on correlations. Using this theory, between and/or within-sample hypotheses about equality of correlations can be tested. Much of the theory presented in this paper may be described as a straightforward complement to work by Browne (1977, 1984), McDonald (1975), Steiger (1980b), and Steiger and Hakstian (1982). However, the *Mathematica* software implementation in WBCORR presents a simple, unified method for performing the tests on virtually any number of groups with raw data or correlation matrix input. In

addition, researchers may employ WBCORR with *Mathematica's* Monte Carlo sampling capabilities to assess the performance characteristics of these test statistics in any anticipated application.

Table 1

Sample Data Set 1

X_1	X_2	X_3	X_4	X_5	X_6
1.24	2.61	0.26	1.05	7.55	0.20
0.28	2.36	1.19	1.66	0.47	0.46
0.37	0.82	3.23	0.12	4.57	1.84
0.11	5.73	3.04	1.04	0.38	0.35
10.75	2.66	2.62	0.29	4.12	0.38
0.21	0.27	0.42	1.07	1.41	0.24
0.95	0.83	1.00	1.00	1.35	0.90
0.78	0.87	6.99	3.01	2.34	0.41
0.65	0.80	0.70	0.60	2.38	0.49
0.57	0.28	0.55	0.57	0.20	4.60
2.99	0.42	0.34	2.80	0.36	0.67
2.87	1.41	0.49	0.13	3.18	0.53
4.87	2.72	0.39	1.35	1.52	1.89
0.83	2.09	1.00	0.39	0.82	1.80
0.32	3.28	1.91	1.13	1.47	1.44
0.59	5.06	1.07	0.40	3.25	0.57
0.14	0.22	0.25	0.64	0.53	1.07
1.67	1.83	0.85	0.15	0.89	2.39
1.86	0.52	0.14	2.94	1.40	0.97
0.69	0.19	0.46	1.39	4.32	1.51
0.8	4.00	1.90	2.24	2.77	1.31
0.66	0.51	4.52	0.57	4.84	0.82
0.35	0.27	0.78	0.37	0.25	0.72
1.91	1.02	4.32	0.59	1.05	2.35
0.43	0.41	1.67	1.43	0.62	2.64

Table 2

Correlations for 3 Variables on Two Occasions (N=245)

	Time 1			Time 2	
X_1	X_2	X_3	X_4	X_5	X_6
1.00	0.20	0.25	0.44	0.38	0.41
0.20	1.00	0.37	0.33	0.29	0.32
0.25	0.37	1.00	0.29	0.22	0.28
0.44	0.33	0.29	1.00	0.41	0.36
0.38	0.29	0.22	0.41	1.00	0.49
0.41	0.32	0.28	0.36	0.49	1.00

Table 3

Correlations Between 3 Variables in Two Groups[1]

	GPA	SAT-V	SAT-M
GPA	1.00	0.44	0.38
SAT-V	0.31	1.00	0.42
SAT-M	0.29	0.24	1.00

[1] Correlations for Group 1, $N = 521$, above the diagonal. Correlations for Group 2, $N = 644$, below the diagonal.

Appendix 1

Input and Output for Equicorrelation Hypothesis Example.

Part a. Two-stage GLS Analysis

Preliminary Input Lines:

```
In[1]:=  00 wbcorr.m
```

```
In[2]:=  data 1 Import4"EquicorrelationData.csv"8; MatrixForm4data8
```

Out[2]//MatrixForm=

$$
\begin{pmatrix}
1.24 & 2.61 & 0.26 & 1.05 & 7.55 & 0.2 \\
0.28 & 2.36 & 1.19 & 1.66 & 0.47 & 0.46 \\
0.37 & 0.82 & 3.23 & 0.12 & 4.57 & 1.84 \\
0.11 & 5.73 & 3.04 & 1.04 & 0.38 & 0.35 \\
10.75 & 2.66 & 2.62 & 0.29 & 4.12 & 0.38 \\
0.21 & 0.27 & 0.42 & 1.07 & 1.41 & 0.24 \\
0.95 & 0.83 & 1. & 1. & 1.35 & 0.9 \\
0.78 & 0.87 & 6.99 & 3.01 & 2.34 & 0.41 \\
0.65 & 0.8 & 0.7 & 0.6 & 2.38 & 0.49 \\
0.57 & 0.28 & 0.55 & 0.57 & 0.2 & 4.6 \\
2.99 & 0.42 & 0.34 & 2.8 & 0.36 & 0.67 \\
2.87 & 1.41 & 0.49 & 0.13 & 3.18 & 0.53 \\
4.87 & 2.72 & 0.39 & 1.35 & 1.52 & 1.89 \\
0.83 & 2.09 & 1. & 0.39 & 0.82 & 1.8 \\
0.32 & 3.28 & 1.91 & 1.13 & 1.47 & 1.44 \\
0.59 & 5.06 & 1.07 & 0.4 & 3.25 & 0.57 \\
0.14 & 0.22 & 0.25 & 0.64 & 0.53 & 1.07 \\
1.67 & 1.83 & 0.85 & 0.15 & 0.89 & 2.39 \\
1.86 & 0.52 & 0.14 & 2.94 & 1.4 & 0.97 \\
0.69 & 0.19 & 0.46 & 1.39 & 4.32 & 1.51 \\
0.8 & 4. & 1.9 & 2.24 & 2.77 & 1.31 \\
0.66 & 0.51 & 4.52 & 0.57 & 4.84 & 0.82 \\
0.35 & 0.27 & 0.78 & 0.37 & 0.25 & 0.72 \\
1.91 & 1.02 & 4.32 & 0.59 & 1.05 & 2.35 \\
0.43 & 0.41 & 1.67 & 1.43 & 0.62 & 2.64
\end{pmatrix}
$$

```
In[3]:=  equicorrelationHypothesis 1  | 1  2  1  1  0 |
                                       | 1  3  1  1  0 | ;
                                       | 1  3  2  1  0 |
```

Command Line and Output:

In[4]:= **ComputeChiSquare4, data0, equicorrelationHypothesis, , 250,**
   DataType « RawData8

WBCORR Output

Hypothesis Matrix

```
Group     Row      Column     Parameter Tag     Fixed Value
1         2        1          1                 0
1         3        1          1                 0
1         3        2          1                 0
```

Input Correlation Matrix 1   Sample Size 1 25

```
1.          0.108968    0.00694539  !0.0735271  0.195968    !0.116128
0.108968    1.          0.0769198   !0.0665498  0.107224    !0.239101
0.00694539  0.0769198   1.          0.128691    0.150023    !0.0691065
!0.0735271  !0.0665498  0.128691    1.          !0.152004   !0.19305
0.195968    0.107224    0.150023    !0.152004   1.          !0.322708
!0.116128   !0.239101   !0.0691065  !0.19305    !0.322708   1.
```

OLS Estimates of Correlation Matrix 1

```
1.          0.0642778   0.0642778   !0.0735271  0.195968    !0.116128
0.0642778   1.          0.0642778   !0.0665498  0.107224    !0.239101
0.0642778   0.0642778   1.          0.128691    0.150023    !0.0691065
!0.0735271  !0.0665498  0.128691    1.          !0.152004   !0.19305
0.195968    0.107224    0.150023    !0.152004   1.          !0.322708
!0.116128   !0.239101   !0.0691065  !0.19305    !0.322708   1.
```

2! Stage GLS Parameter Estimates

```
Parameter Tag     Estimate       Standard Error
1                 0.0642778      0.124453
```

Significance Test Results

```
Chi Square     df      plevel
0.140485       2       0.932168
```

Assessment of Multivariate Normality

```
      Multivariate Skewness     Chi Square     df        plevel
1     27.3963                   114.151        56        7.3502±10^{16}

      Multivariate Kurtosis     Z              plevel <2!tailed@
1     55.1678                   2.77102        0.00558816
```

Part b. Two-Stage ADF Analysis

Command Line and Output:

In[5]:= **ComputeChiSquare**4, **data**0, **equicorrelationHypothesis,** , **25**0,
    DataType « **RawData, EstimationMethod** « **TSADF** 8

WBCORR Output

Hypothesis Matrix

Group	Row	Column	Parameter Tag	Fixed Value
1	2	1	1	0
1	3	1	1	0
1	3	2	1	0

Input Correlation Matrix 1   Sample Size 1 **25**

$$
\begin{pmatrix}
1. & 0.108968 & 0.00694539 & !0.0735271 & 0.195968 & !0.116128 \\
0.108968 & 1. & 0.0769198 & !0.0665498 & 0.107224 & !0.239101 \\
0.00694539 & 0.0769198 & 1. & 0.128691 & 0.150023 & !0.0691065 \\
!0.0735271 & !0.0665498 & 0.128691 & 1. & !0.152004 & !0.19305 \\
0.195968 & 0.107224 & 0.150023 & !0.152004 & 1. & !0.322708 \\
!0.116128 & !0.239101 & !0.0691065 & !0.19305 & !0.322708 & 1.
\end{pmatrix}
$$

OLS Estimates of Correlation Matrix 1

$$
\begin{pmatrix}
1. & 0.0642778 & 0.0642778 & !0.0735271 & 0.195968 & !0.116128 \\
0.0642778 & 1. & 0.0642778 & !0.0665498 & 0.107224 & !0.239101 \\
0.0642778 & 0.0642778 & 1. & 0.128691 & 0.150023 & !0.0691065 \\
!0.0735271 & !0.0665498 & 0.128691 & 1. & !0.152004 & !0.19305 \\
0.195968 & 0.107224 & 0.150023 & !0.152004 & 1. & !0.322708 \\
!0.116128 & !0.239101 & !0.0691065 & !0.19305 & !0.322708 & 1.
\end{pmatrix}
$$

2! Stage ADF Parameter Estimates

Parameter Tag	Estimate	Standard Error
1	0.0806234	0.0942435

Significance Test Results

Chi Square	df	plevel
0.776573	2	0.678218

Assessment of Multivariate Normality

	Multivariate Skewness	Chi Square	df	plevel
1	27.3963	114.151	56	7.3502 ± 10^{16}

	Multivariate Kurtosis	Z	plevel <2! tailed@	
1	55.1678	2.77102	0.00558816	

Appendix 2

Comparison of 2 Dependent Correlations

Command Statements and Selected Output:

00 **wbcorr.m**

data 1 **Import** 4 **"EquicorrelationData.csv"** 8;

hypothesis 1 > $\begin{matrix} \textbf{1} & \textbf{2} & \textbf{1} & \textbf{1} & \textbf{0} \\ \textbf{1} & \textbf{3} & \textbf{2} & \textbf{1} & \textbf{0} \end{matrix}$ B;

ComputeChiSquare 4, **data** 0, **hypothesis**, , 25 0, **DataType** « **RawData**,
 EstimationMethod « **TSADF** 8

WBCORR Output

Hypothesis Matrix

Group	Row	Column	Parameter Tag	Fixed Value
1	2	1	1	0
1	3	2	1	0

Input Correlation Matrix 1   Sample Size 1 **25**

```
    1.        0.108968   0.00694539  ! 0.0735271   0.195968   ! 0.116128
  0.108968     1.        0.0769198   ! 0.0665498   0.107224   ! 0.239101
 0.00694539  0.0769198    1.          0.128691    0.150023   ! 0.0691065
! 0.0735271  ! 0.0665498  0.128691     1.         ! 0.152004  ! 0.19305
  0.195968    0.107224   0.150023    ! 0.152004    1.         ! 0.322708
! 0.116128   ! 0.239101  ! 0.0691065 ! 0.19305    ! 0.322708   1.
```

OLS Estimates of Correlation Matrix 1

```
    1.        0.092944   0.00694539  ! 0.0735271   0.195968   ! 0.116128
  0.092944     1.        0.092944    ! 0.0665498   0.107224   ! 0.239101
 0.00694539  0.092944     1.          0.128691    0.150023   ! 0.0691065
! 0.0735271  ! 0.0665498  0.128691     1.         ! 0.152004  ! 0.19305
  0.195968    0.107224   0.150023    ! 0.152004    1.         ! 0.322708
! 0.116128   ! 0.239101  ! 0.0691065 ! 0.19305    ! 0.322708   1.
```

2 ! Stage ADF Parameter Estimates

Parameter Tag	Estimate	Standard Error
1	0.0945422	0.0946411

Significance Test Results

Chi Square	df	plevel
0.0190163	1	0.89032

Assessment of Multivariate Normality

	Multivariate Skewness	Chi Square	df	plevel
1	27.3963	114.151	56	$7.3502 \pm 10^{!6}$

	Multivariate Kurtosis	Z	plevel < 2 ! tailed @
1	55.1678	2.77102	0.00558816

Appendix 3

Perfect Circumplex Hypothesis

Preliminary Input Lines and Random Number Generation:

```
In[1]:= << wbcorr.m

In[2]:= << Statistics`DescriptiveStatistics`

In[3]:= << Statistics`MultinormalDistribution`

In[4]:= << Statistics`MultiDescriptiveStatistics`
```

```
In[5]:= circumplexForm6x6 = {{1, f1, f2, f3, f2, f1},
                             {f1, 1, f1, f2, f3, f2},
                             {f2, f1, 1, f1, f2, f3},
                             {f3, f2, f1, 1, f1, f2},
                             {f2, f3, f2, f1, 1, f1},
                             {f1, f2, f3, f2, f1, 1}};
```

```
In[6]:= populationMatrix = circumplexForm6x6 /. {f3 -> .2, f2 -> .4, f1 -> .6};
        MatrixForm[populationMatrix]
```

```
Out[6]//MatrixForm=
    1    0.6  0.4  0.2  0.4  0.6
    0.6  1    0.6  0.4  0.2  0.4
    0.4  0.6  1    0.6  0.4  0.2
    0.2  0.4  0.6  1    0.6  0.4
    0.4  0.2  0.4  0.6  1    0.6
    0.6  0.4  0.2  0.4  0.6  1
```

```
In[7]:= populationDistribution =
          MultinormalDistribution[{0, 0, 0, 0, 0, 0}, populationMatrix];

In[8]:= SeedRandom[41238]

In[9]:= data = RandomArray[populationDistribution, 500];
```

In[10]:= **R ⎵ CorrelationMatrix⟦data⟧; MatrixForm⟦R⟧**

Out[10]//MatrixForm=

$$
\begin{pmatrix}
1. & 0.598603 & 0.417062 & 0.233987 & 0.415227 & 0.593991 \\
0.598603 & 1. & 0.620032 & 0.417526 & 0.187959 & 0.353934 \\
0.417062 & 0.620032 & 1. & 0.580356 & 0.396385 & 0.199658 \\
0.233987 & 0.417526 & 0.580356 & 1. & 0.616586 & 0.433602 \\
0.415227 & 0.187959 & 0.396385 & 0.616586 & 1. & 0.623676 \\
0.593991 & 0.353934 & 0.199658 & 0.433602 & 0.623676 & 1.
\end{pmatrix}
$$

In[11]:= **circumplexHypothesis ⎵**
$$
\begin{pmatrix}
1 & 2 & 1 & 1 & 0 \\
1 & 3 & 2 & 1 & 0 \\
1 & 4 & 3 & 1 & 0 \\
1 & 5 & 4 & 1 & 0 \\
1 & 6 & 5 & 1 & 0 \\
1 & 3 & 1 & 2 & 0 \\
1 & 4 & 2 & 2 & 0 \\
1 & 5 & 3 & 2 & 0 \\
1 & 6 & 4 & 2 & 0 \\
1 & 4 & 1 & 3 & 0 \\
1 & 5 & 2 & 3 & 0 \\
1 & 6 & 3 & 3 & 0 \\
1 & 5 & 1 & 2 & 0 \\
1 & 6 & 2 & 2 & 0 \\
1 & 6 & 1 & 1 & 0
\end{pmatrix}
$$
;

Command Line and Output:

In[12]:= **ComputeChiSquare⟦ data⟦, circumplexHypothesis, , 500⟧, DataType « RawData⟧**

WBCORR Output

Hypothesis Matrix

Group	Row	Column	Parameter Tag	Fixed Value
1	2	1	1	0
1	3	2	1	0
1	4	3	1	0
1	5	4	1	0
1	6	5	1	0
1	3	1	2	0
1	4	2	2	0
1	5	3	2	0
1	6	4	2	0
1	4	1	3	0
1	5	2	3	0
1	6	3	3	0
1	5	1	2	0
1	6	2	2	0
1	6	1	1	0

Input Correlation Matrix 1   Sample Size 1 **500**

```
⎧   1.      0.598603 0.417062 0.233987 0.415227 0.593991⎫
⎪ 0.598603    1.     0.620032 0.417526 0.187959 0.353934⎪
⎪ 0.417062 0.620032    1.     0.580356 0.396385 0.199658⎪
⎪ 0.233987 0.417526 0.580356    1.     0.616586 0.433602⎪
⎪ 0.415227 0.187959 0.396385 0.616586    1.     0.623676⎪
⎩ 0.593991 0.353934 0.199658 0.433602 0.623676    1.    ⎭
```

OLS Estimates of Correlation Matrix 1

```
⎧   1.      0.605541 0.405623 0.207201 0.405623 0.605541⎫
⎪ 0.605541    1.     0.605541 0.405623 0.207201 0.405623⎪
⎪ 0.405623 0.605541    1.     0.605541 0.405623 0.207201⎪
⎪ 0.207201 0.405623 0.605541    1.     0.605541 0.405623⎪
⎪ 0.405623 0.207201 0.405623 0.605541    1.     0.605541⎪
⎩ 0.605541 0.405623 0.207201 0.405623 0.605541    1.    ⎭
```

$2!$ Stage GLS Parameter Estimates

```
Parameter Tag    Estimate     Standard Error
1                0.605541     0.015424
2                0.405623     0.0245621
3                0.207201     0.0328644
```

Significance Test Results

```
Chi Square      df      plevel
6.82337         12      0.869062
```

Assessment of Multivariate Normality

```
      Multivariate Skewness    Chi Square    df      plevel
1     0.666646                 55.5538       56      0.491696

      Multivariate Kurtosis    Z             plevel <2!tailed@
1     48.3778                  0.649714      0.515877
```

Appendix 4

Equality of Independent Correlation Matrices

Input:

```
In[1]:=  << wbcorr.m
```

```
                  ⎧  1    .44   .38 ⎫
In[2]:= data1 ↰ ⎪ .44   1    .42 ⎪;
                  ⎩ .38   .42   1  ⎭
```

```
In[3]:= N1 ↰ 521;
```

```
                  ⎧  1    .31   .29 ⎫
In[4]:= data2 ↰ ⎪ .31   1    .24 ⎪;
                  ⎩ .29   .24   1  ⎭
```

```
In[5]:= N2 ↰ 644;
```

```
                       ⎧ 1 2 1 1 0 ⎫
                       ⎪ 1 3 1 2 0 ⎪
                       ⎪ 1 3 2 3 0 ⎪
In[6]:= hypothesis ↰ ⎪ 2 2 1 1 0 ⎪;
                       ⎪ 2 3 1 2 0 ⎪
                       ⎩ 2 3 2 3 0 ⎭
```

Command Line and Output:

In[7]:= **ComputeChiSquare**4, **data1, data2**0, **hypothesis**, , **N1, N2**08

WBCORR Output

Hypothesis Matrix

Group	Row	Column	Parameter Tag	Fixed Value
1	2	1	1	0
1	3	1	2	0
1	3	2	3	0
2	2	1	1	0
2	3	1	2	0
2	3	2	3	0

Input Correlation Matrix 1   Sample Size 1 **521**

$$\begin{pmatrix} 1 & 0.44 & 0.38 \\ 0.44 & 1 & 0.42 \\ 0.38 & 0.42 & 1 \end{pmatrix}$$

Input Correlation Matrix 2   Sample Size 1 **644**

$$\begin{pmatrix} 1 & 0.31 & 0.29 \\ 0.31 & 1 & 0.24 \\ 0.29 & 0.24 & 1 \end{pmatrix}$$

OLS Estimates of Correlation Matrix 1

$$\begin{pmatrix} 1 & 0.368126 & 0.330241 \\ 0.368126 & 1 & 0.320482 \\ 0.330241 & 0.320482 & 1 \end{pmatrix}$$

OLS Estimates of Correlation Matrix 2

$$\begin{pmatrix} 1 & 0.368126 & 0.330241 \\ 0.368126 & 1 & 0.320482 \\ 0.330241 & 0.320482 & 1 \end{pmatrix}$$

2! Stage GLS Parameter Estimates

Parameter Tag	Estimate	Standard Error
1	0.368126	0.0253494
2	0.330241	0.0261252
3	0.320482	0.0263114

Significance Test Results

Chi Square	df	plevel
14.5103	3	0.00228681

Appendix 5

Equality of Circumplex Structures in Two Groups

Preliminary Input Lines and Random Number Generation:

```
In[1]:=  << wbcorr.m

In[2]:=  << Statistics`DescriptiveStatistics`

In[3]:=  << Statistics`MultinormalDistribution`

In[4]:=  << Statistics`MultiDescriptiveStatistics`

In[5]:=  << LinearAlgebra`MatrixManipulation`
```

$$
In[6]:=\ \textbf{circumplexForm6x6} = \begin{pmatrix} 1 & f_1 & f_2 & f_3 & f_2 & f_1 \\ f_1 & 1 & f_1 & f_2 & f_3 & f_2 \\ f_2 & f_1 & 1 & f_1 & f_2 & f_3 \\ f_3 & f_2 & f_1 & 1 & f_1 & f_2 \\ f_2 & f_3 & f_2 & f_1 & 1 & f_1 \\ f_1 & f_2 & f_3 & f_2 & f_1 & 1 \end{pmatrix};
$$

```
In[7]:=  populationMatrix = circumplexForm6x6 /. {f_3 -> .2, f_2 -> .4, f_1 -> .6};
         MatrixForm[populationMatrix]
```

Out[7]//MatrixForm=

$$
\begin{pmatrix} 1 & 0.6 & 0.4 & 0.2 & 0.4 & 0.6 \\ 0.6 & 1 & 0.6 & 0.4 & 0.2 & 0.4 \\ 0.4 & 0.6 & 1 & 0.6 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 & 1 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.4 & 0.6 & 1 & 0.6 \\ 0.6 & 0.4 & 0.2 & 0.4 & 0.6 & 1 \end{pmatrix}
$$

```
In[8]:=  populationDistribution =
             MultinormalDistribution[{0, 0, 0, 0, 0, 0}, populationMatrix];

In[9]:=  SeedRandom[41238]

In[10]:= data1 = RandomArray[populationDistribution, 500];

In[11]:= data2 = RandomArray[populationDistribution, 250];
```

Command Line and Output:

In[15]:= **ComputeChiSquare**4, **data1, data2**O, **circumplexHypothesis,**
    , **500, 250**O, **DataType** « **RawData**8

WBCORR Output

Hypothesis Matrix

Group	Row	Column	Parameter Tag	Fixed Value
1	2	1	1	0
1	3	2	1	0
1	4	3	1	0
1	5	4	1	0
1	6	5	1	0
1	3	1	2	0
1	4	2	2	0
1	5	3	2	0
1	6	4	2	0
1	4	1	3	0
1	5	2	3	0
1	6	3	3	0
1	5	1	2	0
1	6	2	2	0
1	6	1	1	0
2	2	1	1	0
2	3	2	1	0
2	4	3	1	0
2	5	4	1	0
2	6	5	1	0
2	3	1	2	0
2	4	2	2	0
2	5	3	2	0
2	6	4	2	0
2	4	1	3	0
2	5	2	3	0
2	6	3	3	0
2	5	1	2	0
2	6	2	2	0
2	6	1	1	0

Input Correlation Matrix 1   Sample Size 1 **500**

```
   1.      0.598603 0.417062 0.233987 0.415227 0.593991
 0.598603    1.      0.620032 0.417526 0.187959 0.353934
 0.417062 0.620032    1.      0.580356 0.396385 0.199658
 0.233987 0.417526 0.580356    1.      0.616586 0.433602
 0.415227 0.187959 0.396385 0.616586    1.      0.623676
 0.593991 0.353934 0.199658 0.433602 0.623676    1.
```

Input Correlation Matrix 2   Sample Size 1 **250**

```
   1.      0.612054 0.516346 0.288556 0.436875 0.556876
 0.612054    1.      0.612947 0.351087 0.166406 0.291171
 0.516346 0.612947    1.      0.529379 0.457793 0.155702
 0.288556 0.351087 0.529379    1.      0.624697 0.387267
 0.436875 0.166406 0.457793 0.624697    1.      0.563182
 0.556876 0.291171 0.155702 0.387267 0.563182    1.
```

OLS Estimates of Correlation Matrix 1

```
  1.       0.5981    0.406    0.205987   0.406    0.5981  m
 0.5981    1.        0.5981   0.406     0.205987  0.406
 0.406     0.5981    1.       0.5981    0.406     0.205987
0.205987   0.406     0.5981   1.        0.5981    0.406
 0.406     0.205987  0.406    0.5981    1.        0.5981
 0.5981    0.406     0.205987 0.406     0.5981    1.      o
```

OLS Estimates of Correlation Matrix 2

```
  1.       0.5981    0.406    0.205987   0.406    0.5981  m
 0.5981    1.        0.5981   0.406     0.205987  0.406
 0.406     0.5981    1.       0.5981    0.406     0.205987
0.205987   0.406     0.5981   1.        0.5981    0.406
 0.406     0.205987  0.406    0.5981    1.        0.5981
 0.5981    0.406     0.205987 0.406     0.5981    1.      o
```

2! Stage GLS Parameter Estimates

```
Parameter Tag    Estimate      Standard Error
1                0.5981        0.0128252
2                0.406         0.0199668
3                0.205987      0.0267544
```

Significance Test Results

```
Chi Square     df      plevel
33.1335        27      0.1927
```

Assessment of Multivariate Normality

```
      Multivariate Skewness    Chi Square     df      plevel
1     0.666646                 55.5538        56      0.491696
2     1.20136                  50.0566        56      0.698181

      Multivariate Kurtosis    Z              plevel <2!tailed@
1     48.3778                  0.649714       0.515877
2     47.7694                  0.122539       0.902472
```

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