The 3 Indeterminacies of Common Factor Analysis

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The 3 Indeterminacies of Common Factor Analysis

Introduction

- This module discusses the three indeterminacies of factor analysis.
- We begin with a discussion of the basic equations of the common factor model.

The Common Factor Model

- The unequivocal support that Spearman sought for his "theory of g" fueled his enthusiasm for the common factor model.
- At the random variable level, the m-factor model states that

$$\mathbf{x} = \boldsymbol{\Lambda}\boldsymbol{\xi} + \boldsymbol{\delta} \tag{1}$$

with

$$E(\boldsymbol{\xi}\boldsymbol{\xi}') = \boldsymbol{\Psi}, \quad E(\boldsymbol{\xi}\boldsymbol{\delta}') = \mathbf{0}, \quad E(\boldsymbol{\delta}\boldsymbol{\delta}') = \mathbf{U}^2 \tag{2}$$

where $\mathbf{U}^2$ is a diagonal, positive-definite matrix, $\boldsymbol{\Lambda}$ is the common factor pattern, $\boldsymbol{\Psi}$ the factor intercorrelation matrix, and $\mathbf{U}^2$ contains the unique variances of the variables on its diagonal.
- If **Psi** is an identity matrix and the factors are uncorrelated, we say that the solution is orthogonal, otherwise it is oblique.
- Generally the diagonal of $\boldsymbol{\Psi}$ is assumed have all 1's, since the common factors, as latent variables, can be standardized to any desired variance. (Why? C.P.)

The Common Factor Model

- The model of Equation 1, along with the appropriate side conditions, is sometimes referred to as the "factor model at the random variable level."
- If this model fits the data, then a simple consequence is

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \mathbf{U}^2 \tag{3}$$

- Equation 3, the "fundamental theorem of factor analysis," allows one to test whether the m-factor model is tenable by examining whether a diagonal positive definite $\mathbf{U}^2$ can be found so that $\boldsymbol{\Sigma} - \mathbf{U}^2$ is Gramian and of rank m.

The Common Factor Model

- There were two elements of the factor model that, if identified, could provide substantial practical benefits.
  1. The "factor pattern," $\mathbf{\Lambda}$, by revealing the regression relationships between the observed variables and the more fundamental factors that generate them, could provide information about the structure of the variables being investigated.
  2. sample equivalent of ξ would provide scores on the factors, which would serve as a purified measure of a vitally important construct.
- If, for example, the factor model fit a set of mental ability tests, one could determine a small set of underlying mental abilities that explain a larger number of tests, and the ratings of the test takers on these fundamental abilities. Indeed, Hart and Spearman (1912) envisioned a virtual factor analytic utopia.

The Common Factor Model

Indeed, so many possibilities suggest themselves that it is difficult to speak freely without seeming too extravagant . . . It seems even possible to anticipate the day when there will be yearly official registration of the "intellectual index," as we will call it, of every child throughout the kingdom . . . The present difficulties of picking out the abler children for more advanced education, and the "mentally defective" children for less advanced, would vanish in the solution of the more general problem of adapting education to all . . . Citizens, instead of choosing their career at almost blind hazard, will undertake just the professions really suited to their capacities. One can even conceive the establishment of a minimum index to qualify for parliamentary vote, and above all for the right to have offspring. (Hart & Spearman, 1912, pp. 78–79)

The Common Factor Model

- Unfortunately, it turned out that there was a hierarchy of indeterminacy problems associated with the factor analysis algebra presented above.
- Rather than discuss the problems in the clear, systematic way that simple accuracy would seem to demand, authors committed to the common factor model have generally omitted at least one, or described them in obscure, misleading cliches.
- I describe them here, and urge the reader to compare my description with treatments of the factor model found in many other texts and references.

The 3 Indeterminacy Problems

Identification of Unique Variances

- Identification of $\mathbf{U}^2$. There may be more than one $\mathbf{U}^2$ that, when subtracted from $\mathbf{\Sigma}$, leaves it Gramian and of rank m. This fact, well known to econometricians, and described with considerable clarity and care by Anderson and Rubin (1956), is not described clearly in several factor analysis texts.
- One reason for the confusion may be that necessary and sufficient conditions for identification of $\mathbf{U}^2$ have never been established, and there are a number of incorrect statements and theorems in the literature.

The 3 Indeterminacy Problems

Identification of Unique Variances

- There are some known conditions when $\mathbf{U}^2$ is not identified (described by Anderson and Rubin).
- For example, $\mathbf{U}^2$ is never identified if either $p = 2$ and $m = 1$, or if $p = 4$ and $m = 2$.
- On the other hand, if the number of variables is sufficiently large relative to the number of factors so that $(p - m)^2 > (p + m)$, then $\mathbf{U}^2$ will almost certainly be identified. *However*, if any column of $\boldsymbol{\Lambda}$ can be rotated into a position where it has only 2 non-zero elements (see discussion of rotation below), then $\mathbf{U}^2$ will not be identified.
- This means that the identification of $\mathbf{U}^2$ can never be determined in purely exploratory factor analysis simply by counting the number of observed variables and the number of factors.

The 3 Indeterminacy Problems

Identification of Unique Variances

Example (Unidentified $\mathbf{U}^2$)

Consider the following correlation matrix:

$$\mathbf{R} = \begin{bmatrix} 1.00 & 0.25 \\ 0.25 & 1.00 \end{bmatrix}$$

Suppose we wish to fit a single common factor model to these data. The model will be of the form

$$\mathbf{R} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} = \boldsymbol{\lambda}\boldsymbol{\lambda}' + \mathbf{U}^2$$

The 3 Indeterminacy Problems

Identification of Unique Variances

Example (Unidentified $\mathbf{U}^2$)

In this case, the model is so simple, we can solve it as a system of simultaneous equations. Specifically, you can show that, for

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix},$$

any λ_1 and λ_2 satisfying

$$\lambda_1 \lambda_2 = r,$$

and also satisfying the side conditions that

$$0 < \lambda_i^2 < 1, \quad i = 1, 2$$

will yield an acceptable solution, with diagonal elements of $\mathbf{U}^2$ given by

$$u_i^2 = 1 - \lambda_i^2$$

The 3 Indeterminacy Problems

Identification of Unique Variances

Example (Unidentified $\mathbf{U}^2$)

So, for example, two acceptable solutions, as you may verify, are

$$\boldsymbol{\lambda} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}, \quad \mathbf{U}^2 = \begin{bmatrix} .75 & 0 \\ 0 & .75 \end{bmatrix}$$

and

$$\boldsymbol{\lambda} = \begin{bmatrix} 3/4 \\ 1/3 \end{bmatrix}, \quad \mathbf{U}^2 = \begin{bmatrix} 7/16 & 0 \\ 0 & 8/9 \end{bmatrix}$$

The 3 Indeterminacy Problems

Rotational Indeterminacy

- Rotational Indeterminacy of $\mathbf{\Lambda}$. Even if $\mathbf{U}^2$ is identified, $\mathbf{\Lambda}$ will not be if $m > 1$.
- Suppose, for example, we require m orthogonal factors. If such a model fits, then infinitely many $\mathbf{\Lambda}$ matrices will satisfy $\mathbf{\Sigma} - \mathbf{U}^2 = \mathbf{\Lambda}\mathbf{\Lambda}'$, since $\mathbf{\Lambda}\mathbf{\Lambda}' = \mathbf{\Lambda_1}\mathbf{\Lambda_1}'$ so long as $\mathbf{\Lambda_1} = \mathbf{\Lambda}\mathbf{T}$, for any orthogonal $\mathbf{T}$.
- If one allows correlated common factors, then even more solutions are possible. Starting from a given $\mathbf{\Lambda}$, such that $\mathbf{x} = \mathbf{\Lambda}\boldsymbol{\xi} + \boldsymbol{\delta}$, we see that it is also true that $\mathbf{x} = \mathbf{\Lambda}_1\boldsymbol{\xi}_1 + \boldsymbol{\delta}$, where $\mathbf{\Lambda}_1 = \mathbf{\Lambda}\mathbf{T}$ (for any nonsingular $\mathbf{T}$) and $\boldsymbol{\xi}_1 = \mathbf{T}^{-1}\boldsymbol{\xi}$.
- Thurstone "solved" this very significant problem with his "simple structure criterion," which was essentially a parsimony principle for choosing a $\mathbf{\Lambda}$ that made the resulting factors easy to interpret.

The 3 Indeterminacy Problems

Rotational Indeterminacy

- Thurstone concluded that the common factor model was most appropriately applied when, for any given observed variable, the model used only the smallest number of parameters (factors) to account for the variance of the variable.
- Simple structure meant that a "good $\mathbf{\Lambda}$" should satisfy the following (in an m-factor orthogonal solution):
  1. Each row of $\mathbf{\Lambda}$ should have at least 1 zero.
  2. Each column of $\mathbf{\Lambda}$ should have at least m zeros.
  3. For every pair of columns of $\mathbf{\Lambda}$, there should be several "nonmatching" zeros, i.e., zeros in different rows.
  4. When 4 or more factors are obtained, each pair of columns should have a large proportion of corresponding zero entries.

The 3 Indeterminacy Problems

Rotational Indeterminacy

- In the early days of factor analysis, rotation of the initial $\mathbf{\Lambda}$ with more than 2 columns to a "best simple structure" $\mathbf{\Lambda_1} = \mathbf{\Lambda T}$ was an art, requiring careful calculation and substantial patience.
- Development of "machine rotation" methods and digital computers elevated factor analysis from the status of an esoteric technique understood and practiced by a gifted elite, to a technique accessible (for use and misuse) to virtually anyone.
- Perhaps lost in the shuffle was the important question of why one would expect to find "simple structure" in many variable systems.

The 3 Indeterminacy Problems

Rotational Indeterminacy

Example (Rotational Indeterminacy)

Suppose you factor analyze 6 tests, 3 of which are supposed to be measures of verbal ability, and 3 of which are supposed to be measures of mathematical ability. You factor analyze the data, and are given an "unrotated factor pattern" that looks like the following.

$$\mathbf{\Lambda} = \begin{bmatrix} .424 & .424 \\ .354 & .354 \\ .283 & .283 \\ .424 & -.424 \\ .354 & -.354 \\ .283 & -.283 \end{bmatrix}$$

The 3 Indeterminacy Problems

Rotational Indeterminacy

Example (Rotational Indeterminacy)

- It looks like all 6 of the tests load on the first factor, which we might think of as a "general intelligence factor," while the 3 verbal tests (in the first 3 rows of the factor pattern) load negatively on the second factor, while the 3 mathematical tests load positively.
- It seems that the second factor is some kind of "mathematically and not verbally inclined" factor!
- Of course, there are, as we mentioned above, infinitely many *other* factor patterns that fit the data as well as this one, i.e., produce the identical product $\boldsymbol{\Lambda\Lambda}'$.
- Simply postmultiply $\boldsymbol{\Lambda}$ by any 2×2 orthogonal matrix $\mathbf{T}$, for example, and you will obtain an alternative $\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}\mathbf{T}$.

The 3 Indeterminacy Problems

Rotational Indeterminacy

Example (Rotational Indeterminacy)

- The family of 2×2 orthogonal matrices is of the form

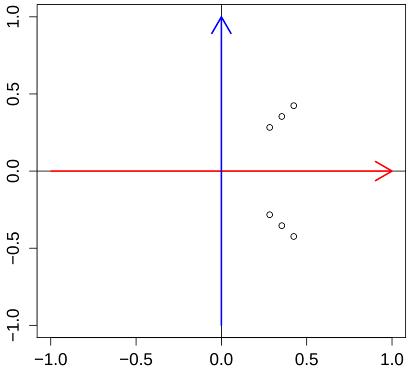
$$\mathbf{T} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

  where θ is the "angle of rotation."
- To see where the term "rotation" comes from, suppose we draw a plot of the 6 variables in "common factor space" by using the factors as our (orthogonal) axes, and the factor loadings as coordinates.
- We obtain a picture as in the figure on the next slide. Note that, in this picture, you can read the factor loadings for any variable by simply reading its coordinates on the two axes.

The 3 Indeterminacy Problems

Rotational Indeterminacy

Example (Rotational Indeterminacy)



The 3 Indeterminacy Problems

Rotational Indeterminacy

Example (Rotational Indeterminacy)

- In this graph, the red vector represents Factor 1, and the blue vector represents Factor 2.
- The 6 observed variables have been projected into the plane spanned by the two common factors.
- The factor loadings are represented by the projection of these 6 points onto the respective factors.
- Now we'll digress with a demonstration hand rotation to simple structure in R.

The 3 Indeterminacy Problems

Factor Indeterminacy

- Factor Indeterminacy. The early factor analysis often spoke as if each factor pattern was associated with a single set of factors.
- It turned out this simply wasn't so.
- If the first two problems are overcome, a third one remains. Specifically, the common and unique factors $\boldsymbol{\xi}$ and $\boldsymbol{\delta}$ are not uniquely defined, even if $\boldsymbol{\Lambda}$ and $\mathbf{U}^2$ are.
- To see this, suppose the factors are orthogonal, and so $\boldsymbol{\Psi} = \mathbf{I}$.

The 3 Indeterminacy Problems

Factor Indeterminacy

- Then consider *any* ξ and δ constructed via the formulas

$$\xi = \Lambda' \Sigma^{-1} \mathbf{x} + \mathbf{P}\mathbf{s} \tag{4}$$

and

$$\delta = \mathbf{U}\Sigma^{-1}\mathbf{x} - \mathbf{U}^{-1}\Lambda\mathbf{P}\mathbf{s} \tag{5}$$

where $\mathbf{s}$ is any arbitrary random vector satisfying

$$E(\mathbf{s}\mathbf{s}') = \mathbf{I} \tag{6}$$

and

$$E(\mathbf{s}\mathbf{x}') = \mathbf{0} \tag{7}$$

$\mathbf{P}$ is an arbitrary Gram-factor satisfying

$$\mathbf{P}\mathbf{P}' = \mathbf{I} - \Lambda'\Sigma^{-1}\Lambda \tag{8}$$

(Students will help derive this formula in class.)

The 3 Indeterminacy Problems

Factor Indeterminacy

- It is straightforward to verify, using matrix expected value algebra, that any ξ and δ satisfying Equations 4–8 will fit the common factor model.
- Once $\boldsymbol{\Lambda}$ is known, $\mathbf{P}$ can be constructed easily via matrix factorization methods. $\mathbf{s}$ is a completely arbitrary random vector in the space orthogonal to that occupied by $\mathbf{x}$.
- Equation 4 shows that common factors are not determinate from the variables in the current analysis.
- There is an infinity of possible candidates for ξ.
- Each has the same "determinate" component $\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\mathbf{x}$, but different "arbitrary component" $\mathbf{Ps}$.
- These candidates for ξ each have the same covariance relationship with $\mathbf{x}$, but possibly differ substantially from each other.

The 3 Indeterminacy Problems

Factor Indeterminacy

- Another way of seeing that factor indeterminacy must exist is to use matrix partitioning to re-express the common factor model.
- Again assume orthogonal factors, and rewrite the factor model as

$$\mathbf{x} = \boldsymbol{\Lambda}\boldsymbol{\xi} + \boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\delta} \end{bmatrix} = \mathbf{B}\mathbf{k} \tag{9}$$

  where $\mathbf{B}$ is a $p \times (p+m)$ matrix.
- In general, $\mathbf{B}$ will have infinitely many orthogonal right unit matrices. An orthogonal right unit is a matrix $\mathbf{T}$ such that $\mathbf{BT} = \mathbf{B}$, $\mathbf{TT}' = \mathbf{I}$. Of course, since $\mathbf{T} \neq \mathbf{I}$, this implies that $\mathbf{T}'\mathbf{k} \neq \mathbf{k}$, and so there are infinitely many different sets of common and unique factors that satisfy the common factor model for the same $\boldsymbol{\Lambda}$ and $\mathbf{U}$.

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

Suppose that the entire population of observations consists of

$$\mathbf{X} = \begin{bmatrix} 0.905 & 1.641 & 0.203 & -1.401 \\ -0.591 & -0.598 & -0.929 & -0.192 \\ -0.501 & 0.370 & 1.848 & 1.752 \\ -0.488 & -0.495 & 0.740 & -0.402 \\ -0.785 & -1.101 & -0.074 & -0.794 \\ -1.598 & 1.216 & -0.404 & -0.900 \\ 0.749 & 0.514 & -1.703 & 1.084 \\ -0.079 & -0.343 & -0.727 & 1.454 \\ 2.132 & 0.576 & 1.226 & -0.001 \\ 0.255 & -1.779 & -0.182 & -0.960 \end{bmatrix}$$

The above matrix may be conceptualized as the entire population of observations, in the sense that each of the 10 row vectors has an equal probability of occurrence. So the matrix represents the full set of outcomes in a discrete multivariate distribution where each of the 10 outcomes has probability of occurrence of $1/10$.

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

In that case, we have

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1.00 & 0.20 & 0.15 & 0.10 \\ 0.20 & 1.00 & 0.12 & 0.08 \\ 0.15 & 0.12 & 1.00 & 0.06 \\ 0.10 & 0.08 & 0.06 & 1.00 \end{bmatrix}, \quad \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 1.066 & -0.191 & -0.132 & -0.083 \\ -0.191 & 1.054 & -0.094 & -0.060 \\ -0.132 & -0.094 & 1.034 & -0.041 \\ -0.083 & -0.060 & -0.041 & 1.016 \end{bmatrix}$$

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

Submitting the above $\boldsymbol{\Sigma}$ to any standard factor analysis program yields the following solutions for $\boldsymbol{\lambda}$ and $\mathbf{U}^2$:

$$\boldsymbol{\lambda} = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.3 \\ 0.2 \end{bmatrix}, \quad \mathbf{U}^2 = \begin{bmatrix} 0.75 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.84 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.91 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.96 \end{bmatrix}$$

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

In order to "construct" a set of common factor scores that agree with the factor model and these data, we need, first of all, to find a component $p\mathbf{s}$ as described in Equations 6–8.

Since there is only one factor, p is a scalar and is equal to the square root of $1 - \boldsymbol{\lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}$. After some tedious calculations, we can determine that $p = 0.775$. Hence, the indeterminate part of any common factor is a deviation score vector $p\mathbf{s}$ such that $\mathbf{X}'\mathbf{s} = \mathbf{0}$, $\mathbf{s}'\mathbf{s}/10 = 1$, and $p = 0.775$.

Infinitely many such vectors exist. To produce one, simply take a vector of random numbers, convert it to deviation score form, multiply it by the complementary projector $\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ to create a vector orthogonal to $\mathbf{X}$, rescale it to the appropriate length, and multiply it by p.

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

Two such candidates for the "indeterminate part" of the common factor are

$$\mathbf{ps}_1 = \begin{bmatrix} 0.398 \\ -0.284 \\ 0.314 \\ 1.949 \\ -0.055 \\ -0.794 \\ 0.608 \\ -0.232 \\ -0.636 \\ -0.640 \end{bmatrix}, \quad \mathbf{ps}_2 = \begin{bmatrix} 0.258 \\ -0.384 \\ 0.509 \\ -0.759 \\ 1.743 \\ -0.515 \\ 0.755 \\ -0.908 \\ -0.261 \\ -0.437 \end{bmatrix}$$

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

The determinate part, also known as the "regression estimates" for the factor scores, is computed directly as

$$\mathbf{X}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} = \begin{bmatrix} 0.742 \\ -0.616 \\ 0.491 \\ -0.241 \\ -0.743 \\ -0.485 \\ 0.245 \\ -0.092 \\ 1.261 \\ -0.563 \end{bmatrix}$$

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

Adding the determinate and indeterminate parts together, we construct two rather different candidates for ξ. They are

$$\xi_1 = \begin{bmatrix} 1.140 \\ -0.900 \\ 0.176 \\ 1.708 \\ -0.798 \\ -1.278 \\ 0.853 \\ -0.323 \\ 0.625 \\ -1.203 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

These candidates for $\boldsymbol{\xi}$ correlate only .399 with each other. It is possible to construct valid candidates for $\boldsymbol{\xi}$ that correlate much less.

Schönemann and Wang (1972) showed that, for orthogonal factors, assuming that $\boldsymbol{\Sigma}$ is a correlation matrix (i.e., that the manifest variables are standardized), the minimum correlation between equivalent factors are given by the diagonal elements of the matrix $2\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda} - \mathbf{I}$.

The 3 Indeterminacy Problems

Factor Indeterminacy

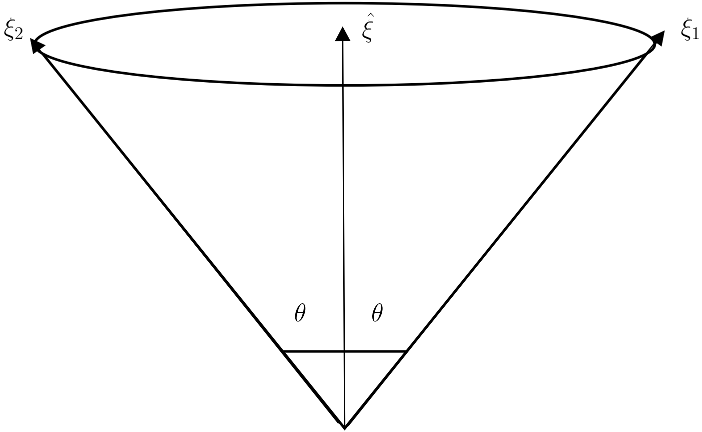


Figure 1: Factor Indeterminacy

The 3 Indeterminacy Problems

Factor Indeterminacy

Example (Factor Indeterminacy)

Imagine that each of the above ξ_i represent the intelligence scores of the individuals manifesting the associated test scores in $\mathbf{X}$. We discover that an individual manifesting score pattern $\mathbf{X}_4' = \begin{bmatrix} -0.488 & -0.495 & 0.740 & -0.402 \end{bmatrix}$ has an intelligence score of 1.708 in one version of the factor, and an intelligence score of -1 in another version. It is this singular fact, first discovered by E. B. Wilson, that seemed to compromise, irretrievably, Spearman's high hopes for measuring g.