# Homework 1

*Instructions.* Feel free to email me for hints if you get stumped. Try to use R as much as possible, but remember that a numerical demonstration of something is not the same as an algebraic proof.

1. (8 points). Given random variables $X$ and $Y$, suppose it is known that both random variables have zero means, and that $E(X^2) = 9$, $E(Y^2) = 4$, and that $E(XY) = 4$. Find the covariance and correlation between $X$ and $Y$, i.e., $\rho_{xy}$ and $\sigma_{xy}$.

2. (20 points). Given the following matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 3 & 2 \\ 3 & 3 & 8 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 3 & 13 \\ 2 & 2 & 4 \\ 3 & 1 & 7 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 6 & 7 & 5 \\ 6 & 8 & 6 \\ 15 & 19 & 11 \end{bmatrix}$$

   Compute the following:

   (a) $\boldsymbol{A} + \boldsymbol{B}$
   (b) $\boldsymbol{CC'}$
   (c) $\boldsymbol{C} - \boldsymbol{3A}$
   (d) $Tr\boldsymbol{AA'}$
   (e) $Tr\boldsymbol{A'C}$

3. Suppose you have a vector $\boldsymbol{x}$, and that $\boldsymbol{x}$ has at least one nonzero element. Define $\boldsymbol{P_x} = \boldsymbol{x}(\boldsymbol{x'x})^{-1}\boldsymbol{x'}$. Note that $\boldsymbol{x'x}$ is a scalar, so $\boldsymbol{P_x}$ may also be written as

$$\boldsymbol{P_x} = \frac{\boldsymbol{xx'}}{\boldsymbol{x'x}} = \left(\frac{1}{\boldsymbol{x'x}}\right)\boldsymbol{xx'}$$

   $\boldsymbol{P_x}$ is known as the "orthogonal projector for $\boldsymbol{x}$." Define $\boldsymbol{Q_x} = \boldsymbol{I} - \boldsymbol{P_x}$.

   Given the above definitions, answer the following:

   (a) Prove that $\boldsymbol{P_x}$ is idempotent by showing that $\boldsymbol{P_x} = \boldsymbol{P_x^2}$.
   (b) Prove that $\boldsymbol{P_x}$ is symmetric by showing that $\boldsymbol{P_x} = \boldsymbol{P_x'}$.
   (c) Prove that $\boldsymbol{Q_x} = \boldsymbol{I} - \boldsymbol{P_x}$ is also symmetric and idempotent.
   (d) Consider any other non-null vector $\boldsymbol{y}$. Prove that $\boldsymbol{a} = \boldsymbol{P_x y}$ is collinear with $\boldsymbol{x}$ i.e., that $\boldsymbol{P_x y}$ can be written in the form $c\boldsymbol{x}$ for some scalar $c$.

(e) Consider the vector $\boldsymbol{b} = \boldsymbol{Q_x y}$. Prove that $\boldsymbol{b}$ is orthogonal to $\boldsymbol{a} = \boldsymbol{P_x y}$, i.e., $\boldsymbol{a'b} = \boldsymbol{b'a} = 0$, and that $\boldsymbol{b}$ is also orthogonal to $\boldsymbol{x}$.

(f) Prove that $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{y}$, thus showing that $\boldsymbol{P_x}$ and $\boldsymbol{Q_x}$ can be used to break a vector $\boldsymbol{y}$ into two component vectors, one orthogonal to $\boldsymbol{x}$, one collinear with $\boldsymbol{x}$.

(g) Create a vector of 5 nonzero numbers. Call it $\boldsymbol{x}$. Using this $\boldsymbol{x}$, create $\boldsymbol{P_x}$ and $\boldsymbol{Q_x}$, create another vector $\boldsymbol{y}$, and then use $\boldsymbol{P_x}$ and $\boldsymbol{Q_x}$ to decompose $\boldsymbol{y}$ into two orthogonal components (one of which is colinear with $\boldsymbol{x}$) that sum to $\boldsymbol{y}$.

4. Consider a data matrix $\boldsymbol{X}$. Prove that if the columns of $\boldsymbol{X}$ are in deviation score form, any linear combination of the variables in $\boldsymbol{X}$, i.e., $\boldsymbol{y} = \boldsymbol{Xb}$ , will also be in deviation score form.

5. An *orthogonal* square matrix $\boldsymbol{X}$ is one for which $\boldsymbol{XX'} = \boldsymbol{X'X} = \boldsymbol{I}$. Suppose that $\boldsymbol{A}$ is symmetric and idempotent. Prove that $\boldsymbol{I} - 2\boldsymbol{A}$ is orthogonal.

6. Suppose random variables $X$ and $Y$ are in standard score form, and have a correlation of $\rho = 0.5$. Let $W = (X + Y)/2$ and $V = X - Y$. Find the following quantities:

   (a) The mean and variance of $W$.

   (b) The variance of $V$.

   (c) The correlation between $V$ and $W$.