The Laws of Linear Combination

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Goals for This Module

In this module, we cover

- What is a linear combination? Basic definitions and terminology
- Key aspects of the behavior of linear combinations
- The Mean of a linear combination
- The Variance of a linear combination
- The Covariance between two linear combinations
- The Correlation between two linear combinations
- Applications

What is a Linear Combination?

An Example: Course Grades

A linear combination of two variables X and Y is any variable C of the form below, where a and b are constants called *linear weights*

$$C = aX + bY \tag{1}$$

The ith score C_i on variable C is produced from the ith scores on X and Y via the formula

$$C_i = aX_i + bY_i \tag{2}$$

I sometimes refer loosely to the above two formulas as the "linear combination rule in "variable form" and the "linear combination rule in score form," respectively.

The constants a and b are called *linear weights*.

What is a Linear Combination?

An Example: Course Grades

Let's consider a classic example:

Consider a course, in which there are two exams, a midterm and a final.

The exams are not "weighted equally." The final exam "counts twice as much" as the midterm.

The course grade for the ith individual is produced with the following equation

$$G_i = \frac{1}{3}X_i + \frac{2}{3}Y_i \tag{3}$$

where X_i is the midterm grade and Y_i the final exam grade.

We say that "G is a linear combination of X and Y with linear weights $1/3$ and $2/3$."

What is a Linear Combination?

An Example: Course Grades

We can think of a linear combination as a recipe that combines "ingredients" to produce a particular result.

For a given set of variables, the linear combination is defined by the linear weights.

Suppose we have two lists of numbers, X and Y. Below is a table of some common linear combination.

X	Y	Name
$+1$	$+1$	Sum
$+1$	-1	Difference
$+\frac{1}{2}$	$+\frac{1}{2}$	Average

Learning to "read" a Linear Combination

It is important to be able to examine an expression and determine the following:

- Is the expression a LC?
- What quantities are being linearly combined?
- What are the linear weights?

For example, consider the following expression. This expression shows how to compute the overall mean $\overline{X}_{\bullet\bullet}$ that results when two samples of size n_1 and n_2, with sample means $\overline{X}_{\bullet 1}$ and $\overline{X}_{\bullet 2}$, are combined into one group of size $n_1 + n_2$.

$$\overline{X}_{\bullet\bullet} = \frac{n_1 \overline{X}_{\bullet 1} + n_2 \overline{X}_{\bullet 2}}{n_1 + n_2} \tag{4}$$

Learning to "read" a Linear Combination

The expression is a linear combination of $\overline{X}_{\bullet 1}$ and $\overline{X}_{\bullet 2}$.

If you look carefully, you can see that the expression can be rewritten as

$$\overline{X}_{\bullet\bullet} = \left(\frac{n_1}{n_1 + n_2}\right)\overline{X}_{\bullet 1} + \left(\frac{n_2}{n_1 + n_2}\right)\overline{X}_{\bullet 2} \quad (5)$$

This is an expression of the form

$$\overline{X}_{\bullet\bullet} = p_1\overline{X}_{\bullet 1} + p_2\overline{X}_{\bullet 2} \quad (6)$$

in which p_1 is the proportion of the total sample size that is from the first sample, and p_2 is the proportion of the total size that is from the second sample. Naturally, these two proportions sum to 1.

Learning to "read" a Linear Combination
The Sample Mean as a Linear Combination

Consider the sample mean

$$\overline{X}_\bullet = \frac{\sum_{i=1}^{n} X_i}{n} \tag{7}$$

The sample mean can be written in the form

$$\overline{X}_\bullet = \sum_{i=1}^{n} \left(\frac{1}{n}\right) X_i \tag{8}$$

In that form, it is easier to see that the sample mean is a linear combination of all the X_i in which all the linear weights are $1/n$.

The Mean of a Linear Combination

When we linearly combine two or more variables, the mean and variance of the resulting linear combination has a known relationship to the mean and variance of the original variables.

Consider the following data, in which final grades G_i for 4 students are computed as

$$G_i = (1/3)X_i + (2/3)Y_i$$

.

The Mean of a Linear Combination

Student	Midterm(X)	Final(Y)	Grade(G)
A	90	78	82
B	82	88	86
C	74	86	82
D	90	96	94
Mean	84	87	86

Notice that the mean for G can be computed from the means of the midterm (X) and final (Y) using the same linear weights used to produce the G_i, namely

$$\begin{aligned}\overline{G}_\bullet &= (1/3)\overline{X}_\bullet + (2/3)\overline{Y}_\bullet \\ &= (1/3)84 + (2/3)87 \\ &= 28 + 58 \\ &= 86\end{aligned}$$

The Mean of a Linear Combination

The preceding example demonstrates the *linear combination rule for means*, i.e., the mean of a linear combination is applied by applying the linear weights to the means of the variables that were linearly combined. So, if

$$G_i = aX_i + bY_i \tag{9}$$

then

$$\overline{G}_\bullet = (1/3)\overline{X}_\bullet + (2/3)\overline{Y}_\bullet \tag{10}$$

The Variance of a Linear Combination

Although the rule for the mean of a LC is simple, the rule for the variance is not so simple. Proving this rule is trivial with matrix algebra, but much more challenging with summation algebra.

At this stage, we will present this rule as a "heuristic rule," a procedure that is easy and yields the correct answer.

We will demonstrate the heuristic rule step by step, using, as an example, the simple linear combination $W = X + Y$. We shall derive an expression for the variance of W.

The Variance of a Linear Combination

- Write the linear combination as a rule for variables. In this case, we write

$$X + Y$$

- Algebraically square the previous expression. In this case, we get

$$(X + Y)^2 = X^2 + Y^2 + 2XY$$

- Apply a natural "conversion rule" to the preceding expression. The rule is:
  1. Replace a squared variable by its variance;
  2. Replace the product of two variables by their covariance.
- Applying the conversion rule, we obtain

$$S_W^2 = S_{X+Y}^2 = S_X^2 + S_Y^2 + 2S_{XY}$$

The Variance of a Linear Combination

An Example

As an example, generate an expression for the variance of the linear combination

$$X - 2Y$$

The Variance of a Linear Combination

An Example

$$(X - 2Y)^2 = X^2 + 4Y^2 - 4XY$$

Applying the conversion rule, we get

$$S^2_{X-4Y} = S^2_X + 4S^2_Y - 4S_{XY}$$

Covariance of Two Linear Combinations

It is quite common to have more than one linear combination on the same columns of numbers.

Consider the following example, in which we compute the sum $(X + Y)$ and the difference $(X - Y)$ on two columns of numbers.

X	Y	$X + Y$	$X - Y$
1	3	4	-2
2	1	3	1
3	2	5	1

Covariance of Two Linear Combinations

The two new variable have a covariance, and we can derive an expression for the covariance using a slight modification of the heuristic rule.

In this case, we simply compute the algebraic product of the two linear combinations, then apply the same conversion rule used previously.

$$(X + Y)(X - Y) = X^2 - Y^2$$

$$S_{X+Y,X-Y} = S_X^2 - S_Y^2$$

We see that the covariance between these two linear combinations is not a function of the covariance between the two original variables. Moreover, if X and Y have the same variance, the covariance between $X + Y$ and $X - Y$ is always zero!

The General Heuristic Rule

Theorem (The General Heuristic Rule)

A general rule that allows computation of the variance of any linear combination or transformation, as well as the covariance between any two linear transformations or combinations, is the following:

- *For the variance of a single expression, write the expression, square it, and apply the simple mnemonic conversion rule described below.*
- *For the covariance of any two expressions, write the two expressions, compute their algebraic product, then apply the conversion rule described below.*

The conversion rule is as follows:

- *All constants are carried forward.*
- *If a term has the product of two variables, replace the product with the covariance of the two variables.*
- *If a term has the square of a single variable, replace the squared variable with its variance.*
- *Any term without the product of two variables or the square of a variable is deleted.*

The General Heuristic Rule

Example (The General Heuristic Rule)

Suppose X and Y are random variables, and you compute the following new random variables:

- $W = X - Y$
- $M = 2X + 5$

Construct formulas for

1. σ^2_W
2. σ^2_M
3. $\sigma_{W,M}$

(Answers on next slide . . .)

The General Heuristic Rule

Example (The General Heuristic Rule)

Answers.

1. To get σ_W^2, we square $X - Y$, obtaining $X^2 + Y^2 - 2XY$, and apply the conversion rule to get $\sigma_W^2 = \sigma_X^2 + \sigma_Y^2 - 2\sigma_{X,Y}$.
2. To get σ_M^2, we square $2X + 5$, obtaining $4X^2 + 20X + 25$. Applying the conversion rule, we drop the last two terms, neither of which have the square of a variable or the product of two variables. We are left with the first term, which yields $\sigma_M^2 = 4\sigma_X^2$.
3. To get $\sigma_{W,M}$, we begin by computing $(X - Y)(2X + 5) = 2X^2 - 2XY + 5X - 5Y$. We drop the last two terms, and obtain $\sigma_{W,M} = 2\sigma_X^2 - 2\sigma_{X,Y}$.