The classic *t* statistic for testing a contrast, in the case of equal *n* is

$$t = \frac{\sum_{j=1}^{a} c_j \overline{Y}_{\bullet j}}{\sqrt{\frac{\sum_{j=1}^{a} c_j^2}{n} MS_{S|A}}} = \frac{\sqrt{n}\sum_{j=1}^{a} c_j \overline{Y}_{\bullet j}}{\sqrt{\sum_{j=1}^{a} c_j^2}\sqrt{MS_{S|A}}} = \frac{\sqrt{n}\sum_{j=1}^{a} c_j^* \overline{Y}_{\bullet j}}{\sqrt{MS_{S|A}}} \tag{1.1}$$

where

$$c_j^* = \frac{c_j}{\sqrt{\sum_{j=1}^{a} c_j^2}}$$

What happens to a set of numbers when we divide them by the square root of their sum of squares? (C.P.)

Note that the c_j^* must also sum to 0.

Now, let us ask ourselves the following question. Given a set of data, what is the maximum possible value of the *t* statistic for a contrast? Notice that any contrast is defined by the linear weights, and so this question really boils down to the following: What set of numbers that sums to zero and has a sum of squares equal to 1 produces the largest possible value of *t*?

Looking more carefully, we see that this in turn devolves into the question, what set of numbers that sums to zero and has a sum of squares equal to 1 produces the largest possible value of

$$c_j^* \overline{Y}_{\bullet j}$$

However, this question, on close inspection, becomes very simple. First, recall something we learned early in Psychology 310. That is

$$\sum (X_i - \overline{X})(Y_i - \overline{Y}) = \sum (X_i)(Y_i - \overline{Y}) = \sum (X_i - \overline{X})(Y_i) \tag{1.2}$$

In other words, the sum of products of deviation scores can be computed with only one of the two variables in deviation score form.

Recall also that for fixed sample size and for a fixed metric of *X* and *Y* scores, the sum of cross-products is proportional to the covariance and the correlation. And so, since the c_j^* are in deviation score form, the set of them that maximizes the value of the *t* statistic is the set of numbers that are perfectly correlated with the sample means, and have a mean of zero, and have a sum of squares equal to 1. Suppose we start with $b_j = (\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet})$. These numbers obviously have a mean of zero, and are obviously correlated 1 with the list of sample means. How do we transform them so that they have a variance of 1? We learned in Psychology 310 that we need simply divide them by their current standard deviation.

Once we transform them so that they have a variance of 1, what is their sum of squares? Since they are already in deviation score form, their sum of squares is equal to $(a-1)S_{\overline{Y}}^2$.

That's right!! So if this is their sum of squares, what do we need to divide them by so that their sum of squares is equal to 1? The answer, as we discovered above, is that we need to divide them by the square root of their current sum of squares, or $\sqrt{a-1}S_{\overline{Y}}$.

So our maximum *t* may be written

$$t_{max} = \frac{\sqrt{n}\sum_{j=1}^{a}(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet})\overline{Y}_{\bullet j}}{S_{\overline{Y}.}\sqrt{(a-1)}\sqrt{MS_{S|A}}} = \frac{\sqrt{n}\sum_{j=1}^{a}(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet})(\overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet})}{S_{\overline{Y}.}\sqrt{(a-1)}\sqrt{MS_{S|A}}} \tag{1.3}$$

Do you see how I got the term on the right from the preceding term?

$$t_{\max} = \frac{\sqrt{n}(a-1)S_{\overline{Y}.}^2}{\sqrt{a-1}S_{\overline{Y}.}\sqrt{MS_{S|A}}} = \frac{\sqrt{n}\sqrt{(a-1)}S_{\overline{Y}.}}{\sqrt{MS_{S|A}}} \tag{1.4}$$

Now, it is well-known that if we square the *t*, we get an *F* statistic for testing the contrast.

So,

$$F_{max} = \frac{(a-1)nS_{\overline{Y}.}^2}{MS_{S|A}} = (a-1)F_{ANOVA}$$

So it turns out, the maximum *F* for any contrast is exactly equal to $(a-1)$ times the *F* statistic you calculated for the 1-way ANOVA on the same data! What does that imply for testing all possible contrasts?

Consider the case in which all group means are the same. Then any contrast has a population value of zero, so all members of the family of contrasts have a true null hypothesis. So the Familywise Error Rate can be controlled at α by controlling the probability that the largest contrast will be rejected at or below α .

Consider the *F* test of the null hypothesis in ANOVA. It is performed by comparing the F_{ANOVA} statistic to a critical value F^*. We have

$$\begin{aligned}
&\Pr(F_{ANOVA} > F^*) = \alpha \\
&\Pr\left[(a-1)F_{ANOVA} > (a-1)F^*\right] = \alpha \\
&\Pr\left[F_{max} > (a-1)F^*\right] = \alpha
\end{aligned} \tag{1.5}$$

We have shown that the probability that the largest possible F statistic one can obtain from any member of the family of all contrasts is controlled at α, and so the probability that any other contrast will reject is smaller than α, *provided that the new critical value*

$$(a-1)F^*_{1-\alpha,(a-1),n_{\bullet}-a} \tag{1.6}$$

is used to perform the F tests. If t tests are computed, the square root of the above value may be used.