Conditional Distributions and the Bivariate Normal Distribution

James H. Steiger

Overview

- In this module, we have several goals:
- Introduce several technical terms
  - Bivariate frequency distribution
  - Marginal distribution
  - Conditional distribution
- In connection with the bivariate normal distribution, discuss
  - Conditional distribution calculations
  - Regression toward the mean

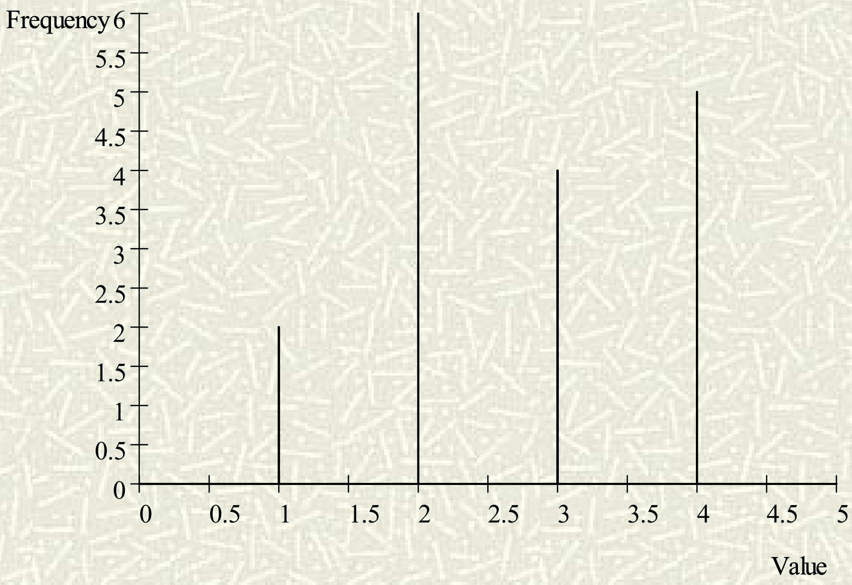
Bivariate Frequency Distributions

- So far, we have discussed *univariate frequency distributions*, which table or plot a set of values and their frequencies.

X	***f***
4	5
3	4
2	6
1	2

Bivariate Frequency Distributions

- We need two dimensions to table or plot a univariate (1-variable) frequency distribution – one dimension for the value, one dimension for its frequency.



Bivariate Frequency Distributions

- A bivariate frequency distribution presents in a table or plot *pairs of values on two variables and their frequencies.*

Bivariate Frequency Distributions

- For example, suppose you throw two coins, *X* and *Y*, simultaneously and record the outcome as an ordered pair of values. Imagine that you threw the coin 8 times, and observed the following (1=Head, 0 = Tail)

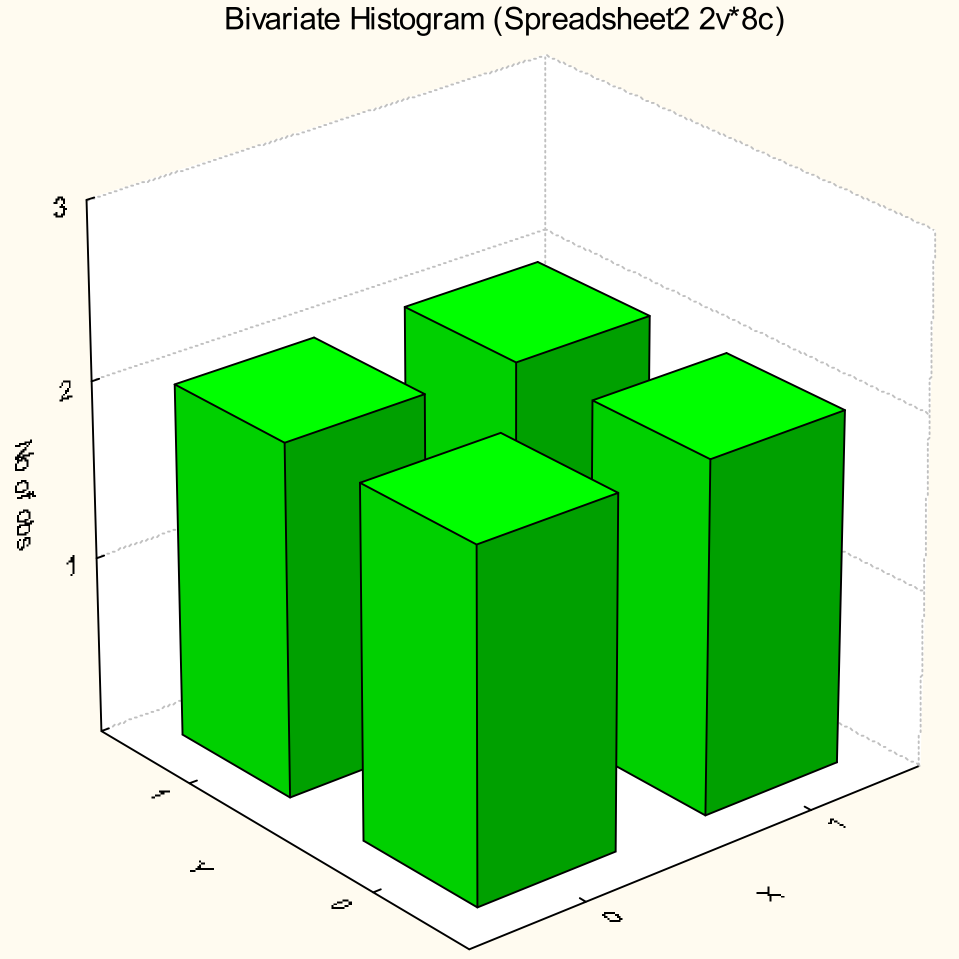
(*X*,*Y*)	***f***
(1,1)	2
(1,0)	2
(0,1)	2
(0,0)	2

Bivariate Frequency Distributions

- To graph the bivariate distribution, you need a 3 dimensional plot, although this can be drawn in perspective in 2 dimensions

(*X*,*Y*)	***f***
(1,1)	2
(1,0)	2
(0,1)	2
(0,0)	2

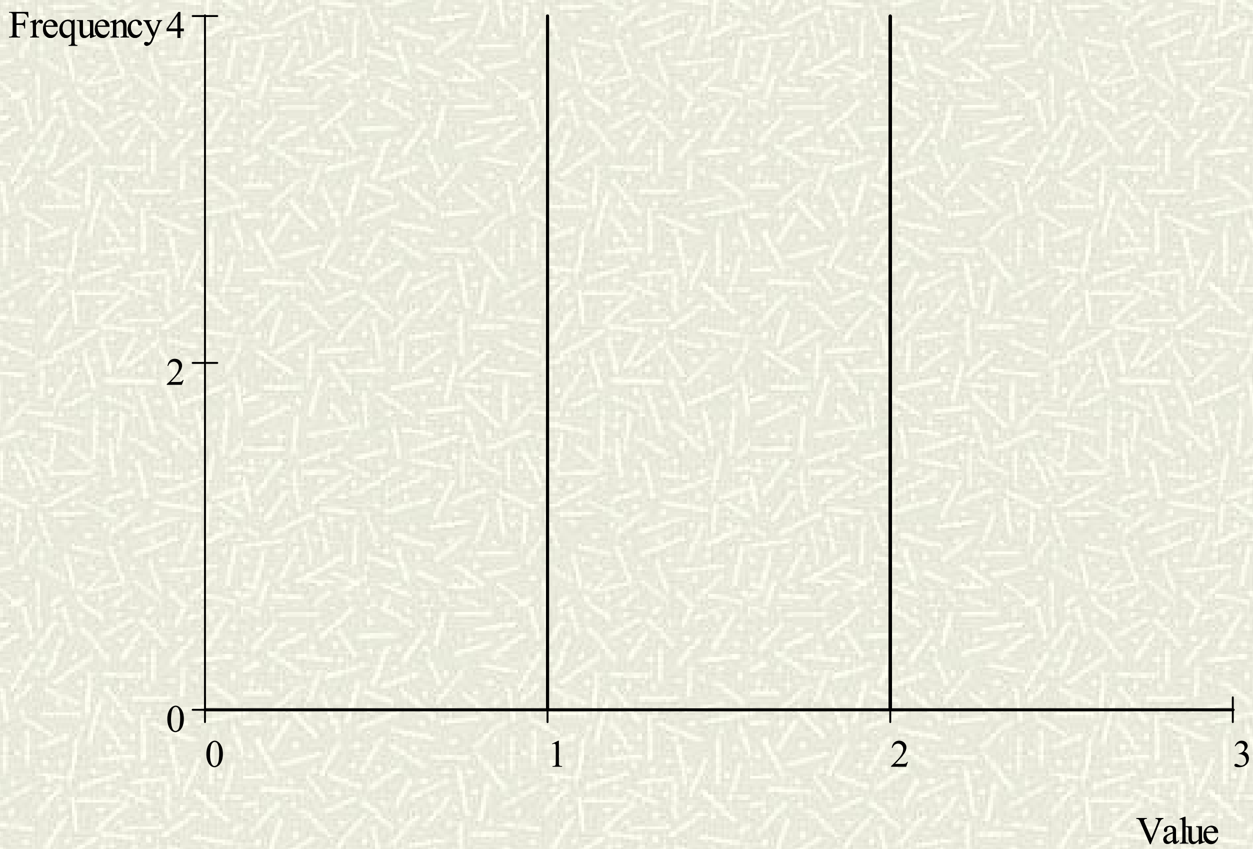
Bivariate Frequency Distributions



Marginal Distributions

- The bivariate distribution of X and Y shows how they behave together. We may also be interested in how X and Y behave separately. We can obtain this information for either X or Y by collapsing (summing) over the opposite variable. For example:

Marginal Distribution of Y



Conditional Distributions

- The *conditional distribution of Y given X=a* is the distribution of *Y* for only those occasions when *X* takes on the value *a*.
- *Example:* The conditional distribution of *Y given X=1* is obtained by extracting from the bivariate distribution only those pairs of scores where *X=1*, then tabulating the frequency distribution of *Y* on those occasions.

Conditional Distributions

- The conditional distribution of *Y given X=1* is:

$Y \mid X = 1$	f
1	2
0	2

Conditional Distributions

- While marginal distributions are obtained from the bivariate by summing, conditional distributions are obtained by "making a cut" through the bivariate distribution.

Marginal, Conditional, and Bivariate Relative Frequencies

- The notion of relative frequency generalizes easily to bivariate, marginal, and conditional probability distributions. In all cases, the frequencies are rescaled by dividing by the total number of observations *in the current distribution table.*

Bivariate Continuous Probability Distributions

- With continuous distributions, we plot *probability density*. In this case, the resulting plot looks like a mountainous terrain, as probability density is registered on a third axis. The most famous bivariate continuous probability distribution is the *bivariate normal.* Glass and Hopkins discuss the properties of this distribution in some detail.

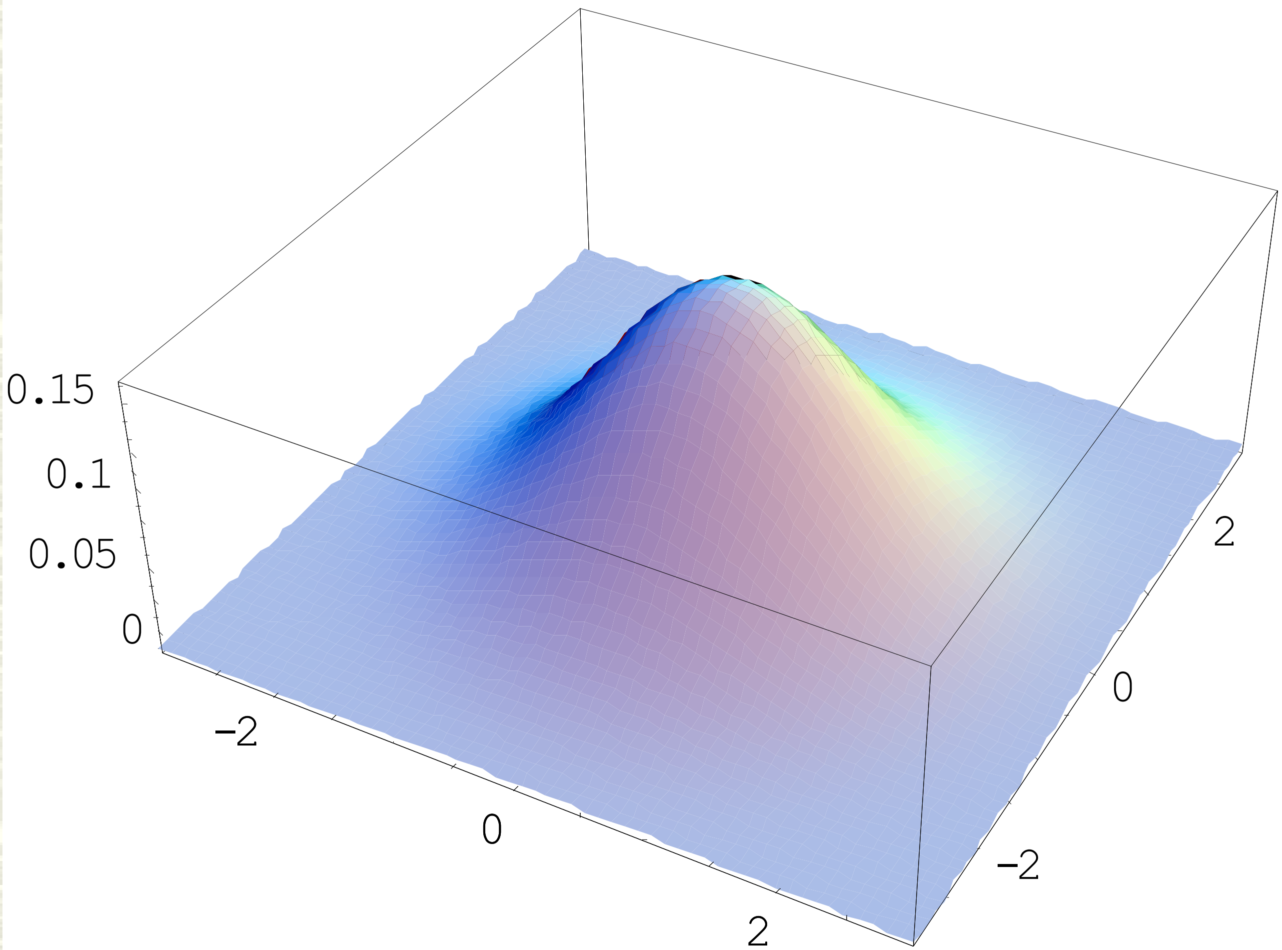
Bivariate Continuous Probability Distributions

- Characteristics of the Bivariate Normal Distribution
  - Marginal Distributions are normal
  - Conditional Distributions are normal, with constant variance for any conditional value.
  - Let *b* and *c* be the slope and intercept of the linear regression line for predicting *Y* from *X*.

$$\mu_{Y|X=a} = ba + c$$

$$\sigma^2_{Y|X=a} = (1 - \rho^2_{YX})\sigma^2_Y = \sigma^2_E$$

The Bivariate Normal Distribution



Computing Conditional Distributions

- We simply use the formulas

$$\mu_{Y|X=a} = ba + c$$

$$\sigma^2_{Y|X=a} = (1 - \rho^2_{YX})\sigma^2_Y = \sigma^2_E$$

We have two choices:

Process the entire problem in the original metric, *or*

Process in *Z* score form, then convert to the original metric.

Conditional Distribution Problems

- The distribution of IQ for women (*X*) and their daughters (*Y*) is bivariate normal, with the following characteristics: Both *X* and *Y* have means of 100 and standard deviations of 15. The correlation between *X* and *Y* is .60.

Conditional Distribution Problems

- First we will process in the original metric. Here are some standard questions. What are the linear regression coefficients for predicting Y from X? They are

$$b = \frac{\rho_{YX}\sigma_Y}{\sigma_X} = .60$$

$$c = \mu_Y - b\mu_X = 100 - (.6)(100) = 40$$

Conditional Distribution Problems

- What is the distribution of IQ scores for women whose mothers had an IQ of 145?
- The mean follows the linear regression rule:

$$\mu_{Y|X=145} = .6(145) + 40 = 127$$

- The standard deviation is

$$\sigma_{Y|X=145} = \sqrt{1-\rho_{XY}^2}\,\sigma_Y$$

$$= \sqrt{1-.36}(15) = (.8)(15) = 12$$

Conditional Distribution Problems

- What percentage of daughters of mothers with IQ scores of 145 will have an IQ at least as high as their mother?

Conditional Distribution Problems

- We simply compute the probability of obtaining a score of 145 or higher in a normal distribution with a mean of 127 and a standard deviation of 12. We have:

$$Z_{145} = \frac{145 - 127}{12} = 1.5$$

The area above 1.5 in the standard normal curve is 6.68%.

Conditional Distribution Problems

- What are the social implications of this result?